Vectors & Tensors

Einstein notation

Any repeated index indicates summation. Thus,
\[ a \cdot b = \sum_{i} a_i b_i = a_i b_i \]
\[ \text{Einstein notation} \]
\[ \text{practical guy!!} \]

Examples
\[ |a| = (a_i a_i)^{1/2}. \]
\[ a_j = \sum_{i} a_i \rightarrow \text{change of coordinates} \]
\[ \text{cosine of angles between axis} \ i \ \text{and} \ j. \]

Any entity that behaves like this is defined as a vector.

\[ \bar{a} \cdot \bar{b} = \bar{a}_j \bar{b}_j = l_{ij} a_i \cdot b_j b_p = \]
\[ = l_{ij} b_p a_i b_p = \sum_{p} a_i b_p = a_i b_i \]
\[ = |a| |b| \quad \text{Axes are orthogonal.} \quad \sum_{p} b_p = b_i \]

\[ a \times b = (a_1 e_1 + a_2 e_2 + a_3 e_3) \times (b_1 e_1 + b_2 e_2 + b_3 e_3) \]
\[ = (a_2 b_3 - a_3 b_2) e_1 + (a_3 b_1 - a_1 b_3) e_2 + (a_1 b_2 - a_2 b_1) e_3 \]
\[ = \left| \begin{array}{ccc}
e_1 & e_2 & e_3 \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \end{array} \right| \]
Let

\[ E_{ijk} = \begin{cases} 0 & \text{if any } y_k \text{ are the same} \\ -1 & \text{if } y_k \text{ is odd permutation of 123} \\ +1 & \text{if } y_k \text{ is even perm. of 123} \end{cases} \]

\[ \Rightarrow a \times b = \sum_{i,j,k} E_{ijk} a_i b_j c_k \]

Similarly, the triple scalar product is:

\[ a \cdot (b \times c) = E_{ijk} a_i b_j c_k \]

Finally, the triple vector product is:

\[ a \times (b \times c) = \sum_{i,j,k} E_{ijk} a_i b_j c_k \]

Tensors

Second order tensors behave like this,

\[ \bar{A}_{pq} = \delta_{ip} \delta_{jq} A_{ij} \]

and they can be treated as matrices.

Quotient rule: if \( A_{ij} \) is a set of mure quantities, \( b \) is independent of \( A \) and \( c \) is a vector, then \( A_{ij} b_i = c_i \Rightarrow A_{ij} \) is a tensor.

To prove it, you need to show that \( A_{ij} b_i c_j = A_{ij} \) is a tensor.
\[ \nabla_i = \frac{\partial}{\partial x_i} \implies \nabla_i \Psi = \frac{\partial \Psi}{\partial x_i} \]

\[ \text{gradient} \]

Gradient vector operator

\[ \nabla \Psi \text{ is a vector. Indeed} \]

\[ \frac{\partial \Psi}{\partial x_j} = \frac{\partial \Psi}{\partial x_i} \frac{\partial x_i}{\partial x_j} = \delta_{ij} \frac{\partial \Psi}{\partial x_i} \]

Prove \( \nabla \Psi \) is the direction of maximum increase of \( \Psi \).

\[ \nabla = \sum_i \frac{\partial}{\partial x_i} \]

Let \( \mathbf{a} \) be a small displacement.

\[ \lim_{\mathbf{a} \to 0} \frac{\Psi(x + \mathbf{a} \mathbf{d} r) - \Psi(x)}{\mathbf{d} r} = \frac{\partial \Psi}{\partial \mathbf{u}}. \]

Using Taylor's Theorem.

\[ \frac{\partial \Psi}{\partial \mathbf{u}} = \nabla \Psi \cdot \mathbf{u} \implies \text{The maximum change takes place when } \mathbf{u} \text{ and } \nabla \Psi \text{ are colinear.} \]
Divergence

\[ \nabla \cdot \mathbf{a} = (\nabla \cdot \mathbf{a}) = a_{11} \frac{\partial a_1}{\partial x_1} + a_{22} \frac{\partial a_2}{\partial x_2} + a_{33} \frac{\partial a_3}{\partial x_3} \]

Watch the notation.

Consider a small volume. Let us construct:

\[ \iiint \mathbf{a} \cdot \mathbf{n} \, ds \]

If \( \mathbf{n} \) denotes the normal to \( dx_2 dx_3 \)

\[ \Rightarrow [a_1(x_1+dx_1), x_2, x_3] - a_1(x_1, x_2, x_3) \, dx_2 dx_3 = \]

\[ \frac{\partial a_1}{\partial x_1} \, dx_1 dx_2 dx_3 + O(d^4) = \frac{\partial a_1}{\partial x_1} \, dV + O(d^4) \]

What did I use here? \( \Rightarrow \iiint \mathbf{a} \cdot \mathbf{n} \, ds = (\nabla \cdot \mathbf{a}) \, dV \)

\[ \lim_{dV \to 0} \frac{1}{dV} \iiint \mathbf{a} \cdot \mathbf{n} \, ds = \nabla \cdot \mathbf{a} \]

If \( \mathbf{a} \) is thought as a flux \( \Rightarrow \iiint \mathbf{a} \cdot \mathbf{n} \, ds \) is the net flux out of the volume \( \Rightarrow \) no generation/conservation of the property in the field (conservation).

\[ \nabla \phi = 0 \Rightarrow \phi \text{ is a solenoidal field} \]

\[ \nabla (\nabla \phi) = \nabla^2 \phi = 0 \Rightarrow \phi \text{ is a potential function} \]
In general, consider a plane triangle whose normal is \( \mathbf{n} \)

\[
\oint_{\partial \Omega} \mathbf{a} \cdot d\mathbf{s} = \oint_{\partial \Omega_1} \mathbf{a} \cdot d\mathbf{s} + \oint_{\partial \Omega_2} \mathbf{a} \cdot d\mathbf{s} + \oint_{\partial \Omega_3} \mathbf{a} \cdot d\mathbf{s}
\]

\[
= E_{ij} \int dA \mathbf{n} \cdot \mathbf{a} \mathbf{e}_j
\]

\[
\Rightarrow \lim_{dA \to 0} \left( \frac{1}{dA} \oint_{\partial \Omega} \mathbf{a} \cdot d\mathbf{s} \right) = (\text{curl} \mathbf{a}) \mathbf{n}
\]

which can be generalized for any curve. (How?)

\( \nabla \times \mathbf{a} = 0 \) \( \Rightarrow \mathbf{a} \) is irrotational

because evidently any circulation around any infinitesimal curve, vanishes.

Note that if \( \mathbf{a} = \nabla \psi \) \( \Rightarrow \)

\( \nabla \times \nabla \psi = 0 \), which may make you think that all irrotational fields can be represented by a scalar function.
Divergence Theorem:

\[ \int_V \nabla \cdot \mathbf{a} \, dV = \int_S \mathbf{a} \cdot \mathbf{n} \, ds \]

which follows directly from our previous findings that for an infinitesimal volume:

\[ \nabla \cdot \mathbf{a} \, dV = \mathbf{a} \cdot \mathbf{n} \, ds \]

Stokes-Kelvin Theorem:

\[ \oint_C \mathbf{a} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{a}) \cdot \mathbf{n} \, ds \]

which follows from previous findings for infinitesimal areas.
Kinematics of Motion

"Path" $\xi(t)$

$x(t)$

$x = x(\xi, t)$

Initial position

actual position

$x$: Spatial coordinates

$\xi$: Material coordinates

Lagrangian or convention are other names

Inverses exist possible $\Rightarrow$

$\exists \exists = \exists (x_1, t)$ exists

which means path is unique! (particles

do not break up and split) or (particles
cannot occupy the same places at the

same time).

$\Rightarrow \ J = \frac{\partial (x_1, x_2, x_3)}{\partial (\xi_1, \xi_2, \xi_3)} \neq 0$
Derivatives

\( \frac{\partial}{\partial t} = (\frac{\partial}{\partial t})_x \)

\( \frac{d}{dt} = (\frac{\partial}{\partial t})_\xi \)

Thus it makes sense to write the velocity of a particle as

\[ \mathbf{v} = \frac{d \mathbf{x}}{dt} \]

In addition, let \( \mathbf{F} = \mathbf{F}(\mathbf{x}, t) \) be a property,

\[ \Rightarrow \quad \frac{d \mathbf{F}}{dt} = \frac{\partial}{\partial t} \mathbf{F}(\mathbf{x}, t) = \frac{\partial}{\partial t} \mathbf{F}(\mathbf{x}(\xi, t), t) \]

\[ = \frac{\partial}{\partial \mathbf{x}_i} (\frac{\partial}{\partial t} \mathbf{x}_i) + \left( \frac{\partial}{\partial t} \mathbf{F} \right)_x \]

\[ \Rightarrow \quad \frac{d \mathbf{F}}{dt} = \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \mathbf{v}_i \frac{\partial \mathbf{F}}{\partial \mathbf{x}_i} \]
Dilatation

\[ dV = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \, d\xi_1 \, d\xi_2 \, d\xi_3 = \mathbf{J} \, dV_0 \]

Now, \( \mathbf{J} = \)

\[
\begin{vmatrix}
\frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\
\frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\
\frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3}
\end{vmatrix}
\]

Let us calculate \( \frac{dJ}{dt} \)

Consider one such term:

\[
\frac{\partial (x_1)}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_1} + \frac{\partial (x_2)}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_1} + \frac{\partial (x_3)}{\partial \xi_1} \frac{\partial x_3}{\partial \xi_1}
\]

Sum of terms:

Each term with the same row dropped.

\[
d \left( \frac{\partial (x_i)}{\partial \xi_j} \right) = \frac{\partial x_i}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_j} = \frac{\partial x_i}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_j}
\]

Expanding the determinant:

One obtains \( \frac{\partial \mathbf{J}}{\partial \xi_1} \)

\[
\Rightarrow \frac{dJ}{dt} = \mathbf{V} \cdot \mathbf{J} \Rightarrow d\ln \mathbf{J} = \mathbf{V} \cdot \frac{d\mathbf{J}}{dt}
\]
Which states why $\nabla \cdot \mathbf{v} = 0$ represents an incompressible fluid.

**Reynolds Transport Theorem**

Let $F(t) = \iiint_{V(t)} F(x,t) \, dV$.

\[
\frac{d}{dt} \int_{V(t)} F(x,t) \, dV = \frac{d}{dt} \int_{V_0} F(x(t,t), t) \, dV_0
\]

\[
= \int_{V_0} \left( \frac{dF}{dt} + \mathbf{F} \nabla \mathbf{v} \right) \, dV_0
\]

What did I do here?

\[
= \int_{V(t)} \left( \frac{dF}{dt} + \mathbf{F} \nabla \mathbf{v} \right) \, dV
\]

and here?

\[
= \int_{V(t)} \left( \frac{dF}{dt} + \nabla (\mathbf{F} \mathbf{v}) \right) \, dV
\]

\[
= \int \left( \frac{\partial F}{\partial t} + \nabla (\mathbf{F} \mathbf{v}) \right) \, dV = \int \frac{\partial F}{\partial t} \, dV
\]

\[
+ \oint_{S(t)} \mathbf{F} \cdot \mathbf{n} \, dS
\]
First consequence:

Conservation of mass and continuity equation:

\[ m = \iiint_V f(x, t) \, dv \]

but \[ \frac{dm}{dt} = 0 \quad \Rightarrow \quad \int_V \left\{ \frac{dp}{dt} + p(\nabla \cdot v) \right\} \, dv = 0 \]

\[ \Rightarrow \frac{dp}{dt} + p \nabla \cdot v = \frac{\partial f}{\partial t} + \nabla (f v) = 0 \]

Continuity equation
\[ \nabla \times \mathbf{a} = \text{curl } \mathbf{a} = \varepsilon_{ijk} a_{k,j} \hat{e}_i \]

Components are

\[
\left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right), \left( \frac{\partial a_1}{\partial x_2} - \frac{\partial a_2}{\partial x_1} \right), \left( \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right)
\]

Let us calculate \( \mathbf{a} \cdot \mathbf{t} \, ds \) on a rectangle.

Consider the vertical sides. The contribution of these sides is:

\[
a_3(x_1, x_2 + dx_2, y_3) - a_3(x_1, x_2, y_3) \, dx_2
\]

\[
= \frac{\partial a_3}{\partial x_2} \, dx_2 \, dx_2 + O(d^3)
\]

\[
\Rightarrow \lim_{{dA \to 0}} \int_{dA} \mathbf{a} \cdot \mathbf{t} \, ds = \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) \, dx_2 \, dx_3
\]

We can show that for a triangle:

\[
\mathbf{a} \cdot ds = \varepsilon_{ijk} \frac{\partial a_i}{\partial x_j} \left( \frac{1}{2} \, ds^2 \cos \theta \right) + O(d^3)
\]

do it. It might be included in the exam.