QUIZ #9 ENGR 3723

PROBLEM 1 (50 points)

A) (35 points) Derive the second order derivative evaluator. $O(h^2)$

ANSWER:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2 + \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h + O(h^2)$$

$$BUT\ \ f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

Therefore, substituting

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2h^2}h + O(h^2)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

B) (15 points) Use it to evaluate the derivative of $f(x) = 3 * \cos(3x - 2)$ at x=0.5. Compare with the result using a first order formula O(h)

ANSWER:

The derivative is : $f'(x) = -3 * \sin(3x - 2)$. Then: $f'(0.5) = -9 * \sin(-0.5) = 4.31483$

We pick h=0.05. Then

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} = \frac{-f(0.6) + 4f(0.55) - 3f(0.5)}{2*0.5} =$$
$$= -3\cos(3*0.6 - 2) + 4*3\cos(3*0.55 - 2) - 3*3\cos(3*0.5 - 2) =$$
$$= 3[-\cos(-0.2) + 4*\cos(-0.35) - 3*\cos(-0.5)]$$
$$= 3[0.98007 + 4*0.93937 - 3*0.87758] = 0.43403$$

Remarkably close!!!!!! For the O(h) case we have

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} = \frac{f(0.55) - f(0.5)}{0.05} = \frac{3\cos(3*0.55 - 2) - 3\cos(3*0.5 - 2)}{0.05} =$$
$$= 60[\cos(-0.35) - \cos(-0.5)] = 60[0.93937 - 0.87758] = 3.70741$$

Not that good!!!

PROBLEM 2 (50 points)

Consider $f(x, y) = x^2 * \cos(y) - 2x$

A) (20 points) Use Univariate search to find the maximum (Start from 1, 1)

ANSWER:

We start with $f(1,1) = \cos(1) - 2 = -1.4597$. We fix ($y$=1) and move in the x direction: $z(x) = x^2 \cos(1) - 2x$. We could do a Golden section search on x, but the answer also comes from $z'(x) = 2x\cos(1) - 2 = 0$, or $x = 1/\cos(1)$ =1.85082. We therefore fix x to this value and proceed, getting $w(y) = 1.85082^2 * \cos(y) - 2*1.85082$. The maximum comes then from: $w'(y) = -1.85082^2 * \sin(y) = 0$, which gives y=0 (we ignore other points). This gives
$f(1.85082,0) = 1.85082^2 \cos(0) - 2*1.85082 = -0.2761$, which is progress.

We stick y=0 in the equation and obtain $z(x) = x^2\cos(0) - 2x = x^2 - 2x$, which by the same procedure as before renders $z'(x) = 2x - 2 = 0$, or x=1.

NOTE: We started from the right number for x, moved away from it and came back!!!

If we stick x=1 in the equation again, we get $w(y) = \cos(y) - 2$ and $w'(y) = -\sin(y) = 0$, which says we do not move and we are done!. This gives $f(1,0) = \cos(0) - 2 = -1$.

Is this a maximum? It can only be determined looking at the Hessian

$$H = \begin{bmatrix} 2*\cos(y) & -2x*\sin(y) \\ -2x*\sin(y) & -x^2*\cos(y) \end{bmatrix} = \begin{bmatrix} 2*\cos(0) & -2*\sin(0) \\ -2*\sin(0) & -\cos(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

The determinant is $|H| = -2$. Then we converged to a saddle point!!!! The univariate method has not worked!!!

B) (30 points) Use Steepest ascent to find the maximum. (Start from 1, 1)

ANSWER:

For this we need to calculate the gradient. Thus,

$$\nabla f(x,y) = \begin{bmatrix} 2x*\cos(y)-2 \\ -x^2*\sin(y) \end{bmatrix}$$

. This is the direction of steepest ascend. Thus,

$$\nabla f(1,1) = \begin{bmatrix} 2*\cos(1)-2 \\ -*\sin(1) \end{bmatrix} = \begin{bmatrix} -0.9194 \\ -0.8415 \end{bmatrix}$$

Thus we set $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-0.9194*h \\ 1-0.8415*h \end{bmatrix}$, which gives:

$$f(x,y) = z(h) = (1-0.9194*h)^2 * \cos(1-0.8415*h) - 2(1-0.9194*h)$$

Thus,

$$z'(h) = 2(-0.9194)(1-0.9194*h)*\cos(1-0.8415*h) - \\ -(-0.8415)(1-0.9194*h)^2 * \sin(1-0.8415*h) - 2(-0.9194)\text{'}$$

which is not easily solvable. The direction to go is for h positive (set by the method)., so we use golden search.

In the quiz I would not expect more than a few iterations of this if not just setting it up.