QUIZ #9

PROBLEM 1 (50 points)

A) (35 points) Derive the second order derivative evaluator. \(O(h^2)\)

**ANSWER:**

\[
\begin{align*}
  f(x_{i+1}) &= f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2 + \cdots \\
  f'(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h + O(h^2) \\
  BUT \ f''(x_i) &= \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} \\
  Therefore, substituting \\
  f'(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2h^2}h + O(h^2) \\
  f''(x_i) &= -\frac{f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)
\end{align*}
\]

B) (15 points) Use it to evaluate the derivative of \( f(x) = 3\cos(3x - 2) \) at \( x=0.5 \). Compare with the result using a first order formula \( O(h) \)

**ANSWER:**

The derivative is : \( f'(x) = -3\sin(3x - 2) \). Then: \( f'(0.5) = -9\sin(-0.5) = 4.31483 \)

We pick \( h=0.05 \). Then

\[
\begin{align*}
  f'(x_i) &= \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} = \frac{-f(0.6) + 4f(0.55) - 3f(0.5)}{2 \times 0.05} \\
  &= -3\cos(3 \times 0.6 - 2) + 4 \times 3\cos(3 \times 0.55 - 2) - 3 \times 3\cos(3 \times 0.5 - 2) \\
  &= 3[\cos(-0.2) + 4\cos(-0.35) - 3\cos(-0.5)] \\
  &= 3[0.98007 + 4 \times 0.93937 - 3 \times 0.87758] = 0.43403
\end{align*}
\]

Remarkably close!!!!!!! For the \( O(h) \) case we have

\[
\begin{align*}
  f'(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{h} = \frac{f(0.55) - f(0.5)}{0.05} = \frac{3\cos(3 \times 0.55 - 2) - 3\cos(3 \times 0.5 - 2)}{0.05} \\
  &= 60[\cos(-0.35) - \cos(-0.5)] = 60[0.93937 - 0.87758] = 3.70741
\end{align*}
\]

Not that good!!!!!
**PROBLEM 2** (50 points)

Consider \( f(x, y) = x^2 \cos(y) - 2x \)

A) (20 points) Use Univariate search to find the maximum (Start from 1, 1)

**ANSWER:**

We start with \( f(1,1) = \cos(1) - 2 = -1.4597 \). We fix \((y=1)\) and move in the x direction: \( z(x) = x^2 \cos(1) - 2x \). We could do a Golden section search on x, but the answer also comes from \( z'(x) = 2x \cos(1) - 2 = 0 \), or \( x = \cos(1) = 1.85082 \). We therefore fix \( x \) to this value and proceed, getting \( w(y) = 1.85082^2 \cos(y) - 2*1.85082 \). The maximum comes then from: \( w'(y) = -1.85082^2 \sin(y) = 0 \), which gives \( y=0 \) (we ignore other points). This gives \( f(1.85082,0) = 1.85082^2 \cos(0) - 2*1.85082 = -0.2761 \), which is progress.

We stick \( y=0 \) in the equation and obtain \( z(x) = x^2 \cos(0) - 2x = x^2 - 2x \), which by the same procedure as before renders \( z'(x) = 2x - 2 = 0 \), or \( x=1 \).

**NOTE:** We started from the right number for x, moved away from it and came back!!!

If we stick \( x=1 \) in the equation again, we get \( w(y) = \cos(y) - 2 \) and \( w'(y) = -\sin(y) = 0 \), which says we do not move and we are done!. This gives \( f(1,0) = \cos(0) - 2 = -1 \).

Is this a maximum? It can only be determined looking at the Hessian

\[
H = \begin{bmatrix}
2 \cos(y) & -2x \sin(y) \\
-2x \sin(y) & x^2 \cos(y)
\end{bmatrix} = \begin{bmatrix}
2 \cos(0) & -2 \sin(0) \\
-2 \sin(0) & -\cos(0)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & 0 \\
0 & -1
\end{bmatrix}
\]

The determinant is \( |H| = -2 \). Then we converged to a saddle point!!!! The univariate method has not worked!!!
B) (30 points) Use Steepest ascent to find the maximum. (Start from 1, 1)

**ANSWER:**

For this we need to calculate the gradient. Thus,

\[
\nabla f(x, y) = \begin{bmatrix}
2x \cdot \cos(y) - 2 \\
-x^2 \cdot \sin(y)
\end{bmatrix}.
\]

This is the direction of steepest ascend. Thus,

\[
\nabla f(1,1) = \begin{bmatrix}
2 \cdot \cos(1) - 2 \\
-\sin(1)
\end{bmatrix} = \begin{bmatrix}
-0.9194 \\
-0.8415
\end{bmatrix}
\]

Thus we set

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
1 - 0.9194 \cdot h \\
1 - 0.8415 \cdot h
\end{bmatrix},
\]

which gives:

\[
f(x, y) = z(h) = (1 - 0.9194 \cdot h)^2 \cdot \cos(1 - 0.8415 \cdot h) - 2(1 - 0.9194 \cdot h)
\]

Thus,

\[
z'(h) = 2(-0.9194)(1 - 0.9194 \cdot h) \cdot \cos(1 - 0.8415 \cdot h)
\]

\[
-(-0.8415)(1 - 0.9194 \cdot h)^2 \cdot \sin(1 - 0.8415 \cdot h) - 2(-0.9194)
\]

which is not easily solvable. The direction to go is for h positive (set by the method), so we use golden search.

In the quiz I would not expect more than a few iterations of this if not just setting it up.