TEST # 3 ENGR 3723

PROBLEM 1 *(10 points)*

Define the coefficient of regression.

ANSWER:

$$r^2 = \frac{S_t - S_r}{S_t}$$

$$S_r = \sum_{\forall i} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2 \qquad S_t = \sum_{\forall i} (y_i - \bar{y})^2$$

PROBLEM 2 *(25 points)*

Consider

x_i	y_i
0	2.1
1	7.7
2	13.6
3	27.2
4	40.9
6	80.3

A) (5 points) Make a linear regression $(y = a_0 + a_1 x)$. Calculate the coefficient of regression.

ANSWER:

$$S_r = \sum_{\forall i} (y_i - a_0 - a_1 x_i)^2$$

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{\forall i}(y_i - a_0 - a_1x_i) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{\forall i}(y_i - a_0 - a_1x_i)x_i = 0$$

Then

$$\sum_{\forall i} y_i = na_0 + a_1\sum_{\forall i} x_i$$

$$\sum_{\forall i} y_i x_i = a_0\sum_{\forall i} x_i + a_1\sum_{\forall i} x_i^2$$

Solve for a_0 and a_1

B) (5 points) Make a second degree polynomial regression. Calculate the coefficient of regression

ANSWER:

$$S_r = \sum_{\forall i}(y_i - a_0 - a_1x_i - a_2x_i^2)^2$$

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{\forall i}(y_i - a_0 - a_1x_i - a_2x_i^2) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{\forall i}(y_i - a_0 - a_1x_i - a_2x_i^2)x_i = 0$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum_{\forall i}(y_i - a_0 - a_1x_i - a_2x_i^2)x_i^2 = 0$$

Then

$$\sum_{\forall i} y_i = na_0 + a_1 \sum_{\forall i} x_i + a_2 \sum_{\forall i} x_i^2$$

$$\sum_{\forall i} y_i x_i = a_0 \sum_{\forall i} x_i + a_1 \sum_{\forall i} x_i^2 + a_2 \sum_{\forall i} x_i^3$$

$$\sum_{\forall i} y_i x_i^2 = a_0 \sum_{\forall i} x_i^2 + a_1 \sum_{\forall i} x_i^3 + a_2 \sum_{\forall i} x_i^4$$

Solve for a_0, a_1 and a_2. See attached file.

C) (10 points) Make a regression using the exponential fit ($y = a_0 e^{a_1 x}$) Calculate the coefficient of regression

ANSWER:

$$\ln y_i = \ln a + bx_i$$

$$then \quad call \quad z_i = \ln y_i$$

and get $z_i = a' + bx_i$. See attached file

D) (5 points) Which regression is the best. Explain why.

ANSWER:

We judge the "goodness of fit" by the values of the coefficient of regression. The second order polynomial is better than the other two (the exponential is actually very bad).

PROBLEM 3 ***(20 points)***

Consider

x_i	y_i
0	2.1
1	7.7
2	13.6
3	27.2
4	40.9
6	80.3

A) (5 points) What is the degree of an interpolating polynomial?

ANSWER:

There are six points. ➔ The interpolating polynomial is of **5th degree**.

B) (7 points) Pick 4 points and obtain the Newton interpolating polynomial.

ANSWER:

Let us pick

x_i	y_i
0	2.1
2	13.6
4	40.9
6	80.3

$$f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2)$$

$$f_n(x) = f(x_0) + (x-x_0)f[x_1, x_0] + (x-x_0)(x-x_1)f[x_2, x_1, x_0]$$

$$+(x-x_0)(x-x_1)x-x_2)f[x_3, x_2, \cdots, x_0]$$

C) (8 points) Pick 4 points and obtain the Lagrange interpolating polynomial.

ANSWER:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \qquad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

In this case:

$$f_3(x)=\sum_{i=0}^{3}L_i(x)f(x_i)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_2)(x_3-x_2)}f(x_3)$$

$$f_3(x)=\frac{(x-2)(x-4)(x-6)}{(0-2)(0-4)(0-6)}2.1+\frac{(x-0)(x-4)(x-6)}{(2-0)(2-4)(2-6)}13.6$$
$$+\frac{(x-0)(x-2)(x-6)}{(4-0)(4-2)(4-6)}40.9+\frac{(x-0)(x-2)(x-4)}{(6-0)(6-2)(6-4)}80.3$$

$$f_3(x)=-0.0438(x-2)(x-4)(x-6)+0.85\,x(x-4)(x-6)-$$
$$-2.5563\,x(x-2)(x-6)+1.6729\,x(x-2)(x-4)$$

<u>PROBLEM 4</u> (25 points)

Consider

x_i	y_i
0	2.1
1	7.7
2	13.6
3	27.2
4	40.9
6	80.3

A) (10 points) Obtain the first order splines through these points.

ANSWER:

$$f_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) = 2.1 + 5.6\ x$$

$$f_2(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = 7.7 + 5.9\ (x - 1)$$

$$f_3(x) = y_2 + \frac{y_3 - y_2}{x_3 - x_2}(x - x_2) = 13.6 + 13.6\ (x - 2)$$

$$f_4(x) = y_3 + \frac{y_4 - y_3}{x_4 - x_3}(x - x_3) = 27.2 + 13.7\ (x - 3)$$

$$f_5(x) = y_4 + \frac{y_5 - y_4}{x_5 - x_4}(x - x_4) = 40.9 + 19.7\ (x - 4)$$

B) (15 points) Obtain the second order splines through these points.

ANSWER:

Equality of splines at interior points (1,2,3,4)

$$a_0\ x_1^2 + b_0\ x_1 + c_0 = y_1$$
$$a_1\ x_1^2 + b_1\ x_1 + c_1 = y_1$$
$$a_1\ x_2^2 + b_1\ x_2 + c_1 = y_2$$
$$a_2\ x_2^2 + b_2\ x_2 + c_2 = y_2$$
$$a_2\ x_3^2 + b_2\ x_3 + c_2 = y_3$$
$$a_3\ x_3^2 + b_3\ x_3 + c_3 = y_3$$
$$a_3\ x_4^2 + b_3\ x_4 + c_3 = y_4$$
$$a_4\ x_4^2 + b_4\ x_4 + c_4 = y_4$$

That is, (we omit a_0 because it is zero by the second derivative =0 at first point)

$$b_0 + c_0 = 7.7$$
$$a_1 + b_1 + c_1 = 7.7$$
$$a_1 \;\; 4 + b_1 \; 2 + c_1 = 13.6$$
$$a_2 \;\; 4 + b_2 \; 2 + c_2 = 13.6$$
$$a_2 \;\; 9 + b_2 \; 3 + c_2 = 27.2$$
$$a_3 \;\; 9 + b_3 \; 3 + c_3 = 27.2$$
$$a_3 \;\; 16 + b_3 \; 4 + c_3 = 40.9$$
$$a_4 \;\; 16 + b_4 \; 4 + c_4 = 40.9$$

Equality of splines at end points (0 and 6)

$$a_0 \;\; x_0^2 + b_0 \; x_0 + c_0 = y_0$$
$$a_4 \;\; x_4^2 + b_4 \; x_4 + c_4 = y_4$$

Then

$$c_0 = 2.1$$
$$a_4 \;\; 36 + b_4 \; 6 + c_4 = 80.3$$

Her we notice that c_0=2.1. Then b_0=5.6.

Equality of first derivatives at interior points (we again consider a_0=0)

$$b_0 = 2a_1 \; x_1 + b_1$$
$$2a_1 x_2 + b_1 = 2a_2 \; x_2 + b_2$$
$$2a_2 x_3 + b_2 = 2a_3 \; x_3 + b_3$$
$$2a_3 x_4 + b_3 = 2a_4 \; x_4 + b_4$$

That is (we used b_0=5.6)

$$5.6 = 2a_1 + b_1$$
$$4a_1 + b_1 = 4a_2 + b_2$$
$$6a_2 + b_2 = 6a_3 + b_3$$
$$8a_3 + b_3 = 8a_4 + b_4$$

Thus the system of equations to solve is:

$$a_1 + b_1 + c_1 = 7.7$$
$$a_1 \;\; 4 + b_1 \;\; 2 + c_1 = 13.6$$
$$a_2 \;\; 4 + b_2 \;\; 2 + c_2 = 13.6$$
$$a_2 \;\; 9 + b_2 \;\; 3 + c_2 = 27.2$$
$$a_3 \;\; 9 + b_3 \;\; 3 + c_3 = 27.2$$
$$a_3 \;\; 16 + b_3 \;\; 4 + c_3 = 40.9$$
$$a_4 \;\; 16 + b_4 \;\; 4 + c_4 = 40.9$$
$$a_4 \;\; 36 + b_4 \;\; 6 + c_4 = 80.3$$
$$5.6 = 2a_1 + b_1$$
$$4a_1 + b_1 = 4a_2 + b_2$$
$$6a_2 + b_2 = 6a_3 + b_3$$
$$8a_3 + b_3 = 8a_4 + b_4$$

We can take the following equations and solve them

$$a_1 + b_1 + c_1 = 7.7$$
$$a_1 \;\; 4 + b_1 \;\; 2 + c_1 = 13.6$$
$$5.6 = 2a_1 + b_1$$

Using Cramer's rule on obtains: a_1=0.3 , b_1=5 , c_1=2.4. Thus, we are left with:

$$a_2 \;\; 4 + b_2 \;\; 2 + c_2 = 13.6$$
$$a_2 \;\; 9 + b_2 \;\; 3 + c_2 = 27.2$$
$$a_3 \;\; 9 + b_3 \;\; 3 + c_3 = 27.2$$
$$a_3 \;\; 16 + b_3 \;\; 4 + c_3 = 40.9$$
$$a_4 \;\; 16 + b_4 \;\; 4 + c_4 = 40.9$$
$$a_4 \;\; 36 + b_4 \;\; 6 + c_4 = 80.3$$
$$6.2 = 4a_2 + b_2$$
$$6a_2 + b_2 = 6a_3 + b_3$$
$$8a_3 + b_3 = 8a_4 + b_4$$

Thus, one can solve:

$$a_2 \;\; 4 + b_2 \;\; 2 + c_2 = 13.6$$
$$a_2 \;\; 9 + b_2 \;\; 3 + c_2 = 27.2$$
$$6.2 = 4a_2 + b_2$$

Using Cramer's rule on obtains: a_2=7.4, b_2=-23.4 , c_2=30.8. Thus, we are left with:

$$a_3 \; 9 + b_3 \; 3 + c_3 = 27.2$$
$$a_3 \; 16 + b_3 \; 4 + c_3 = 40.9$$
$$a_4 \; 16 + b_4 \; 4 + c_4 = 40.9$$
$$a_4 \; 36 + b_4 \; 6 + c_4 = 80.3$$
$$67.8 = 6a_3 + b_3$$
$$8a_3 + b_3 = 8a_4 + b_4$$

Thus, one can solve:

$$a_3 \; 9 + b_3 \; 3 + c_3 = 27.2$$
$$a_3 \; 16 + b_3 \; 4 + c_3 = 40.9$$
$$67.8 = 6a_3 + b_3$$

Using Cramer's rule on obtains: a_3=-54.1 , b_3=392.4 , c_3=-663.1. Thus, we are left with:

$$a_4 \; 16 + b_4 \; 4 + c_4 = 40.9$$
$$a_4 \; 36 + b_4 \; 6 + c_4 = 80.3$$
$$-40.4 = 8a_4 + b_4$$

Using Cramer's rule on obtains: a_4=-54.1 , b_4=392.4 , c_4=-663.1. Thus, we are left with:

PROBLEM 5 ***(20 points)***

Consider $f(x) = \cos x + 3x$

A) (5 points) Obtain $\int_0^3 f(x)dx$ using the rectangle formula. Use no less than 10 points.

ANSWER:

Use 10 points. h=03. Functions are evaluated at 0.15, 0.45, etc.(in the middle of the intervals). Thus:

$$I = \sum_{i=1}^{10} f(x_i) = 0.3\big(f(0.15) + f(0.45) + \ldots + f(2.85)\big) = 13.64165$$

B) (7 points) Obtain $\int_0^3 f(x)dx$ using the trapezoidal formula. Use no less than 10 points.

ANSWER:

Use 10 points. h=0.3

$$I = h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \cdots + h\frac{f(x_{n-1})+f(x_n)}{2}$$

Now x_i=0, 0.3, 0.6 …2.7, 3.

$$I = h\frac{f(x_0)+2\sum_{i=1}^{9} f(x_i)+f(x_n)}{2} = 0.3\frac{f(0.3)+2\big(f(0.6)+...+f(2.7)\big)+f(3)}{2} = 13.64006$$

C) (8 points) Obtain $\int_0^3 f(x)dx$ using the Simpson formula. Use no less than 10 points.

ANSWER:

Use 10 points. h=0.3

$$I \cong \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] \qquad h = b - a$$

Now x_i=0, 0.3, 0.6 …2.7, 3.

$$I = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] + \frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right] + \frac{h}{3}\left[f(x_4)+4f(x_5)+f(x_6)\right] +$$

$$+\frac{h}{3}\left[f(x_6)+4f(x_7)+f(x_8)\right] + \frac{h}{3}\left[f(x_8)+4f(x_9)+f(x_{10})\right] =$$

$$= \frac{h}{3}\left[f(x_0)+4\sum_{=1,(odd)}^{9} f(x_i)+2\sum_{=1,(even)}^{9} f(x_i)+f(x_n)\right] = 13.64113$$

Real value

$$I = \int_0^3 (\cos x + 3x)dx = \sin x\ \Big]_0^3 + 3x^2/2\ \Big]_0^3 = \sin\ 3 + 27/2 = 13.64112$$