BERNOULLI EQUATION

First Law for one input and one output stream (no mass accumulation)

$$\frac{d\left[U+M\left\{\frac{v^2}{2}+mgh\right\}\right]}{dt}=\left[\dot{M}\hat{H}_1-\dot{M}\hat{H}_2+\dot{M}\left\{\frac{v_1^2}{2}-\frac{v_2^2}{2}+mg(h_1-h_2)\right\}\right]+\dot{Q}+\dot{W}_s-P\frac{dV}{dt}$$

For a steady state (open system) we have

$$\frac{d\left[U+M\left\{\frac{v^2}{2}+mgh\right\}\right]}{dt}=0 \quad \text{and} \quad \frac{dV}{dt}=0$$

Then, we write

$$0=\left[\dot{M}(\hat{H}_1-\hat{H}_2)+\dot{M}\left\{\frac{v_1^2}{2}-\frac{v_2^2}{2}+mg(h_1-h_2)\right\}\right]+\dot{Q}+\dot{W}_s$$

We now consider a differential device. , so that $(\hat{H}_2-\hat{H}_1)=d\hat{H}$ and so on. Thus,

$$d\hat{H}=-d\left[\frac{v^2}{2}\right]-mgdh+\frac{\delta\dot{Q}}{\dot{M}}+\frac{\delta\dot{W}_s}{\dot{M}}$$

We use H=U+PV to get:

$$d\hat{U}+d(P\hat{V})=-d\left[\frac{v^2}{2}\right]-mgdh+\frac{\delta\dot{Q}}{\dot{M}}+\frac{\delta\dot{W}_s}{\dot{M}}$$

We now replace $d\hat{U}=Td\hat{S}-pd\hat{V}$ and expand $d(P\hat{V})=Pd\hat{V}+\hat{V}dP$ to get:

$$Td\hat{S}-Pd\hat{V}+Pd\hat{V}+\hat{V}dP=-d\left[\frac{v^2}{2}\right]-mgdh+\frac{\delta\dot{Q}}{\dot{M}}+\frac{\delta\dot{W}_s}{\dot{M}}.$$

Thus:

$$Td\hat{S}+\hat{V}dP=-d\left[\frac{v^2}{2}\right]-mgdh+\frac{\delta\dot{Q}}{\dot{M}}+\frac{\delta\dot{W}_s}{\dot{M}}.$$

Now, the entropy balance is:

$$\frac{dS}{dt} = \left[\dot{M}\hat{S}_1 - \dot{M}\hat{S}_2\right] + \frac{\dot{Q}}{T} + \dot{S}_{gen}$$

But steady state requires $\frac{dS}{dt} = 0$, which gives the differential form:

$$Td\hat{S} = \frac{\delta\dot{Q}}{\dot{M}} + \frac{T\dot{S}_{gen}}{\dot{M}}$$

Then

$$\frac{\delta\dot{Q}}{\dot{M}} = Td\hat{S} - \frac{T\dot{S}_{gen}}{\dot{M}}$$

We replace this in the equation above:

$$\cancel{Td\hat{S}} + \hat{V}dP = -d\left[\frac{v^2}{2}\right] - mgdh + \cancel{Td\hat{S}} - \frac{T\dot{S}_{gen}}{\dot{M}} + \frac{\delta\dot{W}_s}{\dot{M}}$$

Rearranging:

$$\frac{\delta\dot{W}_s}{\dot{M}} = \hat{V}dP + d\left[\frac{v^2}{2}\right] + mgdh + \frac{T\dot{S}_{gen}}{\dot{M}}$$

This is the Bernoulli equation in its most general (thermodynamic form) form

When there is no shaft work ($\dot{W}_s$ =0) and the fluids behave ideally (no friction, that is $\dot{S}_{gen}$ =0), we get

$\hat{V}dP + d\left[\frac{v^2}{2}\right] + mgdh = 0$ or integrated: $\int \hat{V}dP + \Delta\left[\frac{v^2}{2}\right] + mg\Delta h = 0$. For liquids $\hat{V}$ is constant (inverse of density), so we get:

$$\hat{V}\Delta P + \Delta\left[\frac{v^2}{2}\right] + mg\Delta h = 0.$$

When there is friction, but no shaft work, we use $\frac{T\dot{S}_{gen}}{\dot{M}} = 2\frac{f(\mathrm{Re})}{D}v^2 dL$, to get:

$$\frac{\delta \dot{W}_s}{\dot{M}} = \hat{V}dP + d\left[\tfrac{v^2}{2}\right] + mgdh + 2\frac{f(\text{Re})}{D}v^2 dL$$

which in integral form for a pipe of constant diameter D, becomes:

$$\frac{\dot{W}_s}{\dot{M}} = \int \hat{V}dP + \Delta\left[\tfrac{v^2}{2}\right] + mg\Delta h + 2\frac{f(\text{Re})}{D}v^2 L$$

which indicates that to move a fluid through a pipe, we need to overcome the frictionless energy difference $\int \hat{V}dP + \Delta\left[\tfrac{v^2}{2}\right] + mg\Delta h$ PLUS the friction part $2\frac{f(\text{Re})}{D}v^2 L$

THUS, THE BERNOULLI EQUATION

$$\hat{V}\Delta P + \Delta\left[\tfrac{v^2}{2}\right] + mg\Delta h = 0$$

IS NOT A SEPARATE LAW.

IT IS NOTHING MORE THAN THE FIRST LAW AND THE ENTROPY BALANCE COMBINED FOR A ***REVERSIBLE*** SYSTEM.

A MECHANICAL ENERGY BALANCE

$$\frac{\dot{W}_s}{\dot{M}} = \int \hat{V}dP + \Delta\left[\tfrac{v^2}{2}\right] + mg\Delta h + 2\frac{f(\text{Re})}{D}v^2 L$$

FOR PIPING SYSTEMS IS THE FIRST LAW AND THE ENTROPY BALANCE COMBINED FOR AN ***IRREVERSIBLE*** SYSTEM.