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New rigorous one-step MILP formulation for heat exchanger network synthesis

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Abstract

In this paper, a rigorous MILP formulation for grass-root design of heat exchanger networks is developed. The methodology does not rely on traditional supertargeting followed by network design steps typical of the Pinch Design Method, nor is a non-linear model based on superstructures, but rather gives cost-optimal solutions in one step. Unlike most models, it considers splitting, non-isothermal mixing and it counts shells/units. The model relies on transportation/transshipment concepts that are complemented with constraints that allow keeping track of flow rate consistency when splitting takes place and with mechanisms to count heat exchanger shells and units. Several examples from the literature were tested, finding that the model usually obtains better solutions. In some cases, the model produced unknown solutions that were not found using superstructure optimization methods, even when the same pattern of matches is used.

Keyword: Heat exchanges networks

1. Model for grass-root synthesis

The problem of designing heat exchanger networks is one of the oldest problems in process synthesis and perhaps the one that has received the largest attention. The reader is referred to recent books (Biegler, Grossmann, & Westerberg, 1997; Seider, Seader, & Lewin, 1999; Shenoy, 1995; Smith, 1995) for the complete background on all the variety of methodologies developed throughout the years. In addition, the reader may consult three reviews on the topic of HENS by Furman and Sahinidis (2002), Gundersen and Naess (1988) and Jezowski (1994a, 1994b).

A well-known pinch design method emerged throughout the years as the easiest response to the challenge. It relies on two steps, energy supertargeting and final network design. Energy supertargeting tries to determine the trade off between energy and area cost before attempting the design. Once this trade off is determined, a single minimum approach temperature (HRAT) is established and a design is performed, by starting to place matches at the pinch and using a tick-off rule (Linnhoff and Hindmarsh, 1983; Smith, 1995). Designs obtained using the pinch design methodology have been shown to be non-optimal. To ameliorate some of the shortcomings of the pinch design method, an alternative minimum temperature difference, the exchanger minimum approach temperature (EMAT) was introduced and used. At the same time, superstructure-like non-linear mathematical programming models started to be proposed. A large variety of methodologies have been developed after these initial approaches using several alternative objective functions in sequential and one step, as well as iterative procedures. All these formulations are thoroughly reviewed by Furman and Sahinidis (2002).

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Nomenclature

Sets

B	$\{(i, j)\|$ more than one heat exchanger is permitted between hot stream i and cold stream $j\}$
C^z	$\{j\|j$ is a cold stream present in zone $z\}$
C^z_n	$\{j\|j$ is a cold stream present in temperature interval n in zone $z\}$
CU^z	$\{j\|j$ is a cooling utility present in zone $z\}$ $(\mathrm{CU}^z \subset C^z)$
H^z	$\{i\|i$ is a hot stream present in zone $z\}$
H^z_m	$\{i\|i$ is a hot stream present in temperature interval m in zone $z\}$
HU^z	$\{i\|i$ is a heating utility present in zone $z\}$ $(\mathrm{HU}^z \subset H^z)$
M^z	$\{m\|m$ is a temperature interval in zone $z\}$
m^0_i	$\{m\|m$ is the starting temperature interval for hot stream $i\}$
m^f_i	$\{m\|m$ is the final temperature interval for hot stream $i\}$
M^z_i	$\{m\|m$ is a temperature interval belonging to zone z, in which hot stream i is present$\}$
n^0_j	$\{m\|m$ is the starting temperature interval for cold stream $j\}$
n^f_j	$\{m\|m$ is the final temperature interval for cold stream $j\}$
N^z_j	$\{n\|n$ is a temperature interval belonging to zone z, in which cold stream j is present$\}$
NI^H	$\{i\|$ non-isothermal mixing is permitted for hot stream $i\}$
NI^C	$\{j\|$ non-isothermal mixing is permitted for cold stream $j\}$
P	$\{(i, j)\|$ a heat exchange match between hot stream i and cold stream j is permitted$\}$
P^H_{im}	$\{i\|$ heat transfer from hot stream i at interval m to cold stream j is permitted$\}$
P^C_{jn}	$\{j\|$ heat transfer from hot stream i to cold stream j at interval n is permitted$\}$
S^H	$\{i\|$ splits are allowed for hot stream $i\}$
S^C	$\{j\|$ splits are allowed for cold stream $j\}$
Z	$\{z\|z$ is a heat transfer zone$\}$

Parameters

$A^z_{ij\max}$	maximum shell area for an exchanger matching hot stream i and cold stream j in zone z
c^A_{ij}	variable cost for a new heat exchanger matching hot stream i and cold stream j
c^F_{ij}	fixed charge cost for a heat exchanger matching hot stream i and cold stream j
c^H_i	cost of heating utility i
c^C_j	cost of cooling utility j
Cp_{im}	heat capacity of hot stream i at temperature interval m
Cp_{jn}	heat capacity of cold stream j at temperature interval n
F_i	flow rate of hot process stream i
F_j	flow rate of cold process stream j
F^U_i	upper bound for the flow rate of heating utility i
F^U_j	upper bound for the flow rate of cooling utility j
h_{jn}	film heat transfer coefficient for cold stream j in interval n
h_{im}	film heat transfer coefficient for hot stream i in interval m
$\Delta H^{z,H}_{im}$	enthalpy change for hot stream i at interval m of zone z
$\Delta H^{z,C}_{jn}$	enthalpy change for cold stream j at interval n of zone z
q^L_{ijm}	lower bound for heat transfer from hot stream i at interval m to cold stream j
q^L_{ijn}	lower bound for heat transfer from hot stream i to cold stream j at interval n
ΔT_i	temperature range of stream i
ΔT_j	temperature range of stream j
T^U_m	upper temperature of interval m
T^L_m	lower temperature of interval m
$\Delta T^{\mathrm{ML}}_{mn}$	mean logarithmic temperature difference between intervals m and n

Variables

A_{ij}^{z}	area for an exchanger matching hot stream i and cold stream j in zone z
$\hat{A}_{ij}^{z,k}$	area of the kth exchanger matching hot stream i and cold stream j in zone z
$G_{ijm}^{z,k}$	auxiliary binary variable that determines whether the k-th exchanger between hot stream i with cold stream j in zone z exists at interval m of when $(i, j) \in B$.
$K_{ijm}^{z,H}$	determines the beginning of a heat exchanger at interval m of zone z for hot stream i with cold stream j. Defined as binary when $(i, j) \in B$ and as continuous when $(i, j) \notin B$.
$K_{ijn}^{z,C}$	determines the beginning of a heat exchanger at interval n of zone z for cold stream j with hot stream i. Defined as binary when $(i, j) \in B$ and as continuous when $(i, j) \notin B$.
$\hat{K}_{ijm}^{z,H}$	determines the end of a heat exchanger at interval m of zone z for hot stream i with cold stream j. Defined as binary when $(i, j) \in B$ and as continuous when $(i, j) \notin B$.
$\hat{K}_{ijn}^{z,C}$	determines the end of a heat exchanger at interval n of zone z for cold stream j with hot stream i. Defined as binary when $(i, j) \in B$ and as continuous when $(i, j) \notin B$.
$q_{im,jn}^{z}$	heat transfer from hot stream i at interval m to cold stream j at interval n in zone z
$\bar{q}_{inm}^{z,H}$	non-isothermal mixing heat transfer for hot stream i between intervals m and n in zone z
$\bar{q}_{jmn}^{z,C}$	non-isothermal mixing heat transfer for hot stream i between intervals m and n in zone z
$\hat{q}_{ijm}^{z,H}$	heat transfer from hot stream i at interval m to cold stream j in zone z
$\hat{q}_{ijn}^{z,C}$	heat transfer to cold stream j at interval n from hot stream j in zone z
$\tilde{q}_{ijm}^{z,H}$	auxiliary continuous variable utilized to compute the hot side heat load of each heat exchanger when several exchangers exist between hot stream i and cold stream j in zone z
$\tilde{q}_{ijn}^{z,C}$	auxiliary continuous variable utilized to compute the cold side heat load of each heat exchanger when several exchangers exist between hot stream i and cold stream j in zone z
$\breve{q}_{im,jn}^{z}$	auxiliary continuous variable utilized to compute the area of individual heat exchangers between hot stream i with cold stream j in zone z when $(i, j) \in B$.
U_{ij}^{z}	number of shells in the heat exchanger between hot stream i and cold stream j in zone z
$U_{ij}^{z,k}$	number of shells in the kth heat exchanger between hot stream i and cold stream j in zone z
$X_{im,jn}^{z}$	auxiliary continuous variable equal to zero when an exchanger ends at interval m for hot stream i and at interval n for cold stream j. A value of one corresponds to all other cases.
$Y_{ijm}^{z,H}$	determines whether heat is being transferred from hot stream i at interval m to cold stream j. Defined as binary when $(i, j) \notin B$ and as continuous when $(i, j) \in B$.
$Y_{ijn}^{z,C}$	determines whether heat is being transferred from hot stream i to cold stream j at interval n. Defined as binary when $(i, j) \notin B$ and as continuous when $(i, j) \in B$.
$\alpha_{ijm}^{z,H}$	auxiliary continuous variable equal to one when heat transfer from interval m of hot stream i to cold stream j occurs in zone z and it does not correspond to the beginning nor the ending of a heat exchanger. A value of zero corresponds to all other cases.
$\alpha_{ijn}^{z,C}$	auxiliary continuous variable equal to one when heat transfer from hot stream i to interval n of cold stream j occurs in zone z and it does not correspond to the beginning nor the ending of a heat exchanger. A value of zero corresponds to all other cases.

Furman and Sahinidis (2002) discuss the “strong need for the development of approximation algorithms”. This stems from the realization that heat exchanger network design is an NP-Hard problem (Furman & Sahinidis, 2001). They also suggest that the simplifying assumptions that have been used (“isothermal mixing, no split stream following through more than one exchanger and no stream bypass”) diminish the merits of some successful one-step methods. They call for a “truly complete formulation of the HENS problem without any simplifying assumptions.” Some efforts in this direction have been made by Jezowski, Shethna, & Castillo (2003), who proposed linear models. We believe that we are responding to that challenge to a good extent, both on the modeling aspect of limiting the simplifying assumptions to a minimum and proposing a MILP formulation that can be attractive from a computational standpoint. This MILP model is based on the transportation–transshipment paradigm and it has the following features:

- counts heat exchangers units and shells;
- approximates the area required for each exchanger unit or shell;

- controls the total number of units;
- implicitly determines flow rates in splits;
- handles non-isothermal mixing;
- identifies bypasses in split situations when convenient;
- controls the temperature approximation (HRAT/EMAT or $\Delta T_{\min}$) when desired;
- can address block-design through the use of zones;
- allows multiple matches between two streams.

All the above features are the result of a special transshipment/transportation scheme that is capable of precisely describing the structure of the network using different sets of binary variables. Consequently, the model has a remarkable ability to produce cost-optimal networks. The one-step structure of the formulation also presents important advantages in terms of user intervention demand and allows achieving a high degree of design flexibility. Contrasting with to the traditional two-step structure (Targeting/Supertargeting and Network Design) of most of the approximate methods, this new formulation directly gives cost-effective solutions at once. Although there have been attempt to establish one-step procedures based on mathematical programming (complete list provided by Furman & Sahinidis, 2002), our proposed procedure does not rely on any the simplifying assumptions used so far. In addition, unlike others, it is MILP and is reasonably fast. Several examples from the literature were tested, finding that the model usually obtains better solutions in terms of cost-optimality. In some cases, the model produced unknown solutions that were not found using superstructure optimization methods, even when the same pattern of matches is used.

2. Mathematical model

2.1. Set definitions

We now proceed to outline the general philosophy of the model. For this purpose, let us define a number of different sets that will be used throughout the model. First, a set of several heat transfer zones is defined, namely $Z=\{z|z$ is a heat transfer zone$\}$.

This set allows the model to handle restrictions imposed by the designer on the heat transfer from certain temperature intervals to others. For instance, if a designer wanted to explore a network using the pinch design method, two zones (above and below the pinch) are defined. On the other hand, if the designer wants to explore a network that minimizes the total cost, even by transferring heat across the process pinch, only one zone is required. Additionally, the use of zones can be used to separate the design in different sub-networks that are not interrelated, simplifying the network and the problem complexity.

Next, the following sets are used to identify hot, cold streams; and heating, cooling utilities.

$H^z=\{i|i$ is a hot stream present in zone $z\}$
$C^z=\{j|j$ is a cold stream present in zone $z\}$
$\mathrm{HU}^z=\{i|i$ is a heating utility present in zone $z\}$ $(\mathrm{HU}^z \subset H^z)$
$\mathrm{CU}^z=\{j|j$ is a cooling utility present in zone $z\}$ $(\mathrm{CU}^z \subset C^z)$

Additionally, several temperature intervals are considered in each zone, in order to perform the heat balances and the area calculations. These intervals, are then sorted such that if $m_1<m_2$ then $T^U_{m_1} > T^U_{m_2}$, where the superscript U indicates the upper limit of the temperature interval. In addition, a shift of $\Delta T_{\min}$ is performed over all cold streams temperatures to guarantee network feasibility. The value of $\Delta T_{\min}$, however, can be set to zero or any small value, which would be equivalent to using an EMAT. The different sets related to the temperature intervals are:

$M^z=\{m|m$ is a temperature interval in zone $z\}$
$M^z_i=\{m|m$ is a temperature interval belonging to zone z, in which hot stream i is present$\}$
$N^z_j=\{n|n$ is a temperature interval belonging to zone z, in which cold stream j is present$\}$
$H^z_m=\{i|i$ is a hot stream present in temperature interval m in zone $z\}$
$C^z_n=\{j|j$ is a cold stream present in temperature interval n in zone $z\}$
$m^0_i=\{m|m$ is the starting temperature interval for hot stream $i\}$
$n^0_j=\{n|n$ is the starting temperature interval for cold stream $j\}$
$m^f_i=\{m|m$ is the final temperature interval for hot stream $i\}$
$n^f_j=\{n|n$ is the final temperature interval for cold stream $j\}$

The model then uses the temperature intervals to perform energy balances and flow balances. Fig. 1 depicts one hot and one cold stream spanning some temperature intervals and exchanging heat. At each interval, the variables $\hat{q}^{z,H}_{ijm}$ account for the

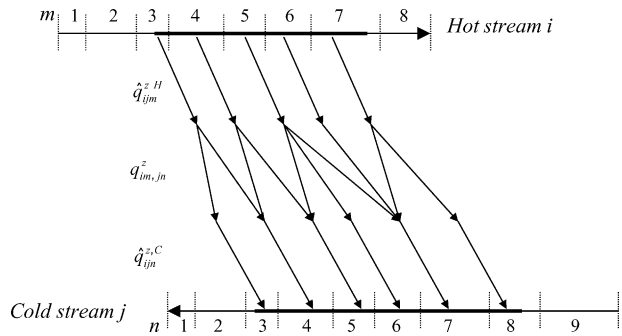


Fig. 1. Basic scheme of the transportation/transshipment model.

overall heat exchanged in interval m of hot stream i and all the intervals of cold stream j, in zone z. Similarly, variables $\hat{q}^{z,C}_{ijn}$ are used to compute the overall heat received by cold stream j at interval n from all intervals of hot stream i. In turn, the variables $\hat{q}^{z,H}_{im,jn}$ variables are used to account for the heat transportation from interval to interval between both streams.

Additionally, a number of sets are introduced to define all possible sources and destinations for heat transfer in this transportation scheme.

$P = \{(i, j)|$heat exchange match between hot stream i and cold stream j is permitted$\}$
$P^H_{im} = \{j|$heat transfer from hot stream i at interval m to cold stream j is permitted$\}$
$P^C_{jn} = \{i|$heat transfer from hot stream i to cold stream j at interval n is permitted$\}$

Set P defines all permitted heat exchange matches between hot and cold streams. In addition to an automatic membership of a pair (i, j) for which exchange is thermodynamically possible, permitted and forbidden heat exchange matches can be set by the designer. In addition, sets P^H_{im} and P^C_{jn} define feasible heat transfer flows at each temperature interval.

Finally, the following sets allow the designer to manage additional features of the formulation, according to his or her own preference.

$\mathrm{NI}^H = \{i|$non-isothermal mixing is permitted for hot stream $i\}$
$\mathrm{NI}^C = \{j|$non-isothermal mixing is permitted for cold stream $j\}$
$S^H = \{i|$splits are allowed for hot stream $i\}$
$S^C = \{j|$splits are allowed for cold stream $j\}$
$B = \{(i, j)|$more than one heat exchanger unit is permitted between hot stream i and cold stream $j\}$

The sets NI^H and NI^C are used to specify whether non-isothermal mixing of stream splits is permitted, while sets S^H and S^C establish the possibility of stream splits. Finally, set B is used to allow more than one heat exchanger match between two streams, as shown in Fig. 2 for match (i_1, j_1). Thus, in contrast to previous formulations this new model is able to distinguish situations where more than one heat exchanger unit is required to perform a heat exchange match.

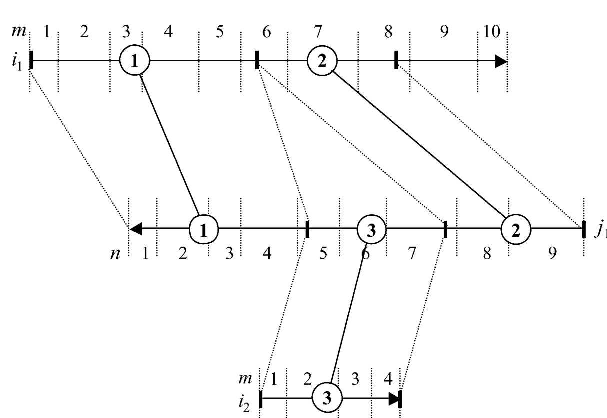


Fig. 2. A case where more than one heat exchanger unit is required for a match (i, j).

Having properly defined the previous collection of sets, we now introduce the different equations of the model for grass-root design of heat exchanger networks.

2.2. Heat balance equations

These equations simply state that the total heat available on each hot streams or the total heat demand of cold streams is equal to the heat transferred to the specific intervals. For heating and cooling utilities, these balances are described by the following equations.

Heat balance for heating utilities:

$$F_i^H\left(T_m^U - T_m^L\right) = \sum_{\substack{n \in M^z \\ T_n^L < T_m^U}} \sum_{\substack{j \in C_n^z \\ J \in P_{im}^H \\ i \in P_{jn}^C}} q_{im,jn}^z \quad z \in Z; m \in M^z; i \in H_m^z; i \in \mathrm{HU}^z \tag{1}$$

Heat balance for cooling utilities:

$$F_j^C\left(T_n^U - T_n^L\right) = \sum_{\substack{n \in M^z \\ T_n^L < T_m^U}} \sum_{\substack{i \in H_m^z \\ i \in P_n^C \\ j \in P_m^H}} q_{im,jn}^z \quad z \in Z; n \in M^z; j \in C_n^z; j \in \mathrm{CU}^z \tag{2}$$

Notice that for utilities the flow rates are considered variable and will be optimally determined by the model. Thus, an a-priori utility targeting stage is not necessary; even though a fixed value for utilities flow rates could still be specified if the designer pleases. In turn, for process streams, the following equations represent the heat balances for cases where only isothermal mixing of splits is considered (non-isothermal mixing is covered later).

Heat balance for hot process streams—$i \notin \mathrm{NI}^H$:

$$\Delta H_{im}^{z,H} = \sum_{\substack{n \in M^z \\ T_n^L < T_m^U}} \sum_{\substack{j \in C_n^z \\ j \in P_{im}^H \\ i \in P_{jn}^C}} q_{im,jn}^z \quad z \in Z; m \in M^z; i \in H_m^z; j \notin \mathrm{HU}^z; i \notin \mathrm{NI}^H \tag{3}$$

Heat balance for cold process streams—$j \notin \mathrm{NI}^C$:

$$\Delta H_{jn}^{z,C} = \sum_{\substack{m \in M^z \\ T_n^L < T_m^U}} \sum_{\substack{i \in H_m^z \\ i \in P_{jn}^C \\ j \in P_{im}^H}} q_{im,jn}^z \quad z \in Z; m \in M^z; j \in C_n^z; j \notin \mathrm{CU}^z; j \notin \mathrm{NI}^C \tag{4}$$

The next sets of equations define the hot and cold cumulative heat transfer. This cumulative transfer is used because it is related to the equations that define the existence of heat exchangers in the different temperature intervals, which are presented later.

Cumulative heat transfer from hot stream i at interval m to cold stream j:

$$\hat{q}_{ijm}^{z,H} = \sum_{\substack{n \in M^z; T_n^L < T_m^U \\ j \in C_n^z; i \in P_{jn}^C}} q_{im,jn}^z \quad z \in Z; m \in M^z; i \in H_m^z; j \in C^z; j \in P_{im}^H \tag{5}$$

Cumulative heat transfer to cold stream j at interval n from hot stream i:

$$\hat{q}_{ijn}^{z,C} = \sum_{\substack{m \in M^z; T_n^L < T_m^U \\ i \in H_m^z; j \in P_{im}^H}} q_{im,jn}^z \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z; i \in P_{jn}^C \tag{6}$$

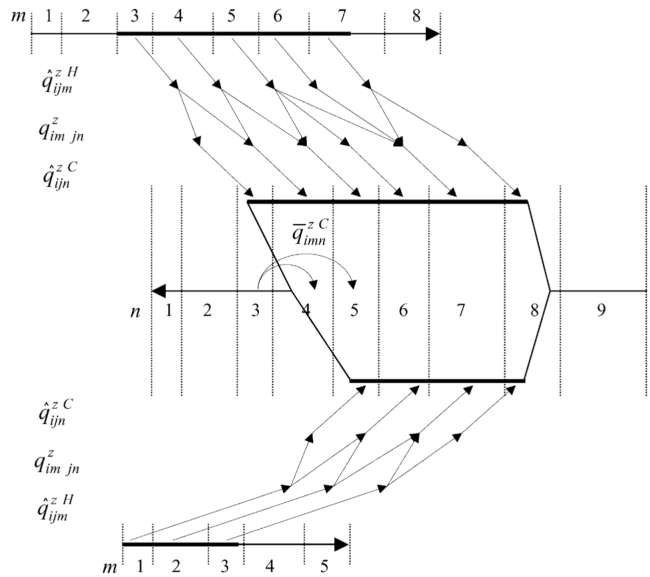


Fig. 3. Non-isothermal split mixing.

2.2.1. Heat balance equations for streams allowed to have non-isothermal split mixing

To consider non-isothermal mixing of stream splits, a new variable ($\overline{q}$) is introduced to account for heat flows between intervals of the same stream that correspond to such mixing. In other words, heat is artificially transferred from one interval to another within the same stream to account for non-isothermal mixing conditions. To illustrate this, consider Fig. 3.

In this case, cold stream j has been split to exchange heat between streams i_1 and i_2 and non-isothermal mixing between these splits is allowed. Notice that the upper portion or the split in the cold stream spans temperature intervals 3 and 8, while the lower portion spans from intervals 5–8. However, after mixing, the whole stream only spans from intervals 4–8, while the non-split part spans the rest of the intervals. To accomplish the non-isothermal mixing, interval 3 receives more heat than its demand ($\Delta H^{z,C}_{j3}$) and transfer this surplus to intervals 4 and 5, as indicated in the figure allowing one branch to reach a larger temperature. In turn, intervals 4 and 5 receive less than their demand from the hot streams, with the difference being transferred from interval 3 by the heat $\overline{q}$. The corresponding heat balance equations are:

Heat balance for hot streams (non-isothermal mixing allowed):

$$\Delta H^{z,H}_{im} = \sum_{\substack{n\in M^z\\ T_n^L<T_m^U}} \sum_{\substack{j\in C_n^z\\ j\in P_{im}^H\\ i\in P_{jn}^C}} q^z_{im,jn} + \sum_{\substack{n\in M^z\\ n>m}} \sum_{i\in H_n^z} \overline{q}^{z,H}_{inm} - \sum_{\substack{n\in M^z\\ n<m}} \sum_{i\in H_n^z} \overline{q}^{z,H}_{imn} \quad z\in Z; m\in M^z; i\in H_m^z; i\notin \mathrm{HU}^z; i\in \mathrm{NI}^H \tag{7}$$

Heat balance for cold streams (non-isothermal mixing allowed):

$$\Delta H^{z,C}_{jn} = \sum_{\substack{m\in M^z\\ T_n^L<T_m^U}} \sum_{\substack{i\in H_m^z\\ i\in P_{jn}^C\\ j\in P_{im}^H}} q^z_{im,jn} + \sum_{\substack{m\in M^z\\ m<n}} \sum_{j\in C_m^z} \overline{q}^{z,C}_{jmn} - \sum_{\substack{m\in M^z\\ m>n}} \sum_{j\in C_m^z} \overline{q}^{z,C}_{inm} \quad z\in Z; m\in M^z; i\in C_n^z; j\notin \mathrm{CU}^z; j\in \mathrm{NI}^C \tag{8}$$

Additionally, constraints enforcing the condition that heat cannot be transferred within a stream unless there exist heat transfer with other stream(s) needs to be introduced in the model. In other words, $\overline{q}$ is forced zero when there is no heat transferred from or to other streams.

Heat balance for hot streams—$i\in \mathrm{NI}^H$:

$$\sum_{\substack{n\in M^z\\ n<m}} \sum_{i\in H_n^z} \overline{q}^{z,H}_{inm} \le \sum_{\substack{n\in M^z\\ T_n^L<T_m^U}} \sum_{\substack{j\in C_n^z; j\in P_{im}^H\\ i\in P_{jn}^C}} q^z_{im,jn} \quad z\in Z; m\in M^z; i\in H_m^z; i\notin \mathrm{HU}^z; i\in \mathrm{NI}^H \tag{9}$$

Heat balance for cold streams—$j \in \mathrm{NI}^C$:

$$\sum_{\substack{m \in M^z \\ m>n}} \sum_{j \in C_m^z} \overline{q}_{inm}^{z,C} \leq \sum_{\substack{m \in M^z \\ T_n^L < T_m^U}} \sum_{\substack{i \in H_m^z; i \in P_{jn}^C \\ j \in P_{im}^H}} q_{im,jn}^z \quad z \in Z; m \in M^z; j \in C_n^z; j \notin \mathrm{CU}^z; j \in \mathrm{NI}^C \tag{10}$$

2.3. Heat exchanger definition and counting

The model considers a heat exchanger is defined as a consecutive series of heat exchange shells between a hot and a cold stream. For each temperature interval, heat transfer is accounted using the cumulative heat ($\hat{q}$), while the existence of a heat exchanger for a given interval is defined by a new variable (Y), which determines whether heat exchange takes place or not at that interval. Additionally, two new variables (K and $\hat{k}$), which are closely related to the Y variables, are introduced in order to indicate whether a heat exchanger begins or ends at a specific interval. The use of these new variables to count units has been previously proposed by Bagajewicz and Rodera (1998) and later used by Bagajewicz and Soto (2001, 2003) and Ji and Bagajewicz (2002).

Multiple shells placed consecutively are treated as a single heat exchanger. Nevertheless, there are cases where non-consecutive series of shells could be allowed. In those cases, different heat exchanger units have to be defined for each series. Therefore, additional equations need to be included to consider the possibility of multiple heat exchangers between the same pair of streams. Set B controls this, as described above.

Consider hot stream i. When only one exchanger is allowed between streams i and j; this is, when $(i, j) \notin B$, then binary variable $Y_{ijm}^{z,H}$ and two continuous variables $K_{ijm}^{z,H}$, $\hat{K}_{ijm}^{z,H}$ are used. The binary variable $Y_{ijm}^{z,H}$ indicates that there is a match between stream i at interval m receiving heat from some intervals of stream j. In turn, $K_{ijm}^{z,H}$ and $\hat{K}_{ijm}^{z,H}$ indicate the beginning and end of a string of intervals for which the binary variable is active. Conversely, when $(i, j) \in B$, $Y_{ijm}^{z,H}$ is declared as continuous and $K_{ijm}^{z,H}$, $\hat{K}_{ijm}^{z,H}$ are declared binary. It will be shown later that in this last case, the Y variables may take a value greater or equal than one if a heat exchanger exists for the correspondent streams and interval. This, however, does not have any effect on the results. Alternatively, a value of zero corresponds to all variables Y_{ijm}, $K_{ijm}^{z,H}$, $\hat{K}_{ijm}^{z,H}$ when no heat exchanger exists matching streams i and j.

The following group of constraints is used to determine the existence of a heat exchanger for a given pair of streams and temperature intervals. Constraints (15)–(19) and (20)–(24) are used when only one heat exchanger is allowed per match. Conversely, Equation (25) applies in cases where more than one exchanger is permitted. Notice also that Equations (15) and (20) only apply to the first and last interval of a hot stream, respectively, while the sets of Equations (16)–(19) and (21)–(24) are used for all intervals.

Bounds on cumulative heat transfer for hot process streams:

$$q_{ijm}^L Y_{ijm}^{z,H} \leq \hat{q}_{ijm}^{z,H} \leq \Delta H_{im}^{z,H} Y_{ijm}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z; i \notin \mathrm{HU}^z; j \in C^z; j \in P_{im}^H \tag{11}$$

Bounds on cumulative heat transfer for cold process streams:

$$q_{ijn}^L Y_{ijn}^{z,C} \leq \hat{q}_{ijn}^{z,C} \leq \Delta H_{jn}^{z,C} Y_{ijn}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z; j \notin \mathrm{CU}^z; i \in P_{jn}^C \tag{12}$$

Bounds on cumulative heat transfer for heating utilities:

$$q_{ijm}^L Y_{ijm}^{z,H} \leq \hat{q}_{ijm}^{z,H} \leq F_i^U (T_m^U - T_m^L) \quad z \in Z; m \in M^z; i \in H_m^z; i \in \mathrm{HU}^z; j \in C^z; j \in P_{im}^H \tag{13}$$

Bounds on cumulative heat transfer for cooling utilities:

$$q_{ijn}^L Y_{ijn}^{z,C} \leq \hat{q}_{ijn}^{z,C} \leq F_j^U (T_n^U - T_n^L) \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z; j \in \mathrm{CU}^z; i \in P_{jn}^C \tag{14}$$

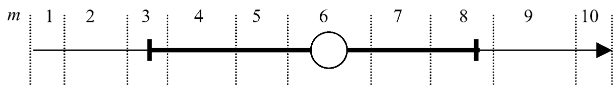
Heat exchanger beginning for hot streams—$(i, j) \notin B$:

$$K_{ijm}^{z,H} \geq Y_{ijm}^{z,H} \quad z \in Z; m \in M^z; m = m_i^0; i \in H^z; j \in C^z; j \in P_{im}^H; (i, j) \notin B \tag{15}$$

$$K_{ijm}^{z,H} \leq 2 - Y_{ijm}^{z,H} - Y_{ijm-1}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in C^z; j \in P_{im}^H \cap P_{im-1}^H; (i, j) \notin B \tag{16}$$

$$K_{ijm}^{z,H} \leq Y_{ijm}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in C^z; j \in P_{im}^H \cap P_{im-1}^H; (i, j) \notin B \tag{17}$$

$$K_{ijm}^{z,H} \geq Y_{ijm}^{z,H} - Y_{ijm-1}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in C^z; j \in P_{im}^H \cap P_{im-1}^H; (i, j) \notin B \tag{18}$$



m	$Y^{z,H}_{ijm}$	$K^{z,H}_{ijm}$	$\hat{K}^{z,H}_{ijm}$
1	0	0	0
2	0	0	0
3	**1**	**1**	0
4	**1**	0	0
5	**1**	0	0
6	**1**	0	0
7	**1**	0	0
8	**1**	0	**1**
9	0	0	0
10	0	0	0

Fig. 4. Heat exchanger definition when $(i, j) \notin B$.

$$K^{z,H}_{ijm} \geq 0 \quad z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m-1}; j \in C^z; j \in P^H_{im} \cap P^H_{im-1}; (i, j) \notin B \tag{19}$$

Heat exchanger ending for hot streams—$(i, j) \notin B$:

$$\hat{K}^{z,H}_{ijm} \geq Y^{z,H}_{ijm} \quad z \in Z; m \in M^z; m = m^f_i; i \in H^z; j \in C^z; j \in P^H_{im}; (i, j) \notin B \tag{20}$$

$$\hat{K}^{z,H}_{ijm} \leq 2 - Y^{z,H}_{ijm} - Y^{z,H}_{ijm+1} \quad z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m+1};\ j \in C^z; j \in P^H_{im} \cap P^H_{im+1}; (i, j) \notin B \tag{21}$$

$$\hat{K}^{z,H}_{ijm} \leq Y^{z,H}_{ijm} \quad z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m+1};\ j \in C^z; j \in P^H_{im} \cap P^H_{im+1}; (i, j) \notin B \tag{22}$$

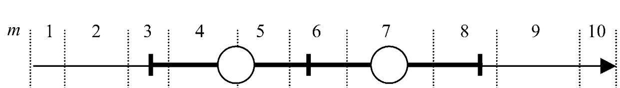
$$\hat{K}^{z,H}_{ijm} \geq Y^{z,H}_{ijm} - Y^{z,H}_{ijm+1} \quad z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m+1};\ j \in C^z; j \in P^H_{im} \cap P^H_{im+1}; (i, j) \notin B \tag{23}$$

$$\hat{K}^{z,H}_{ijm} \geq 0 \quad z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m+1};\ j \in C^z; j \in P^H_{im} \cap P^H_{im+1}; (i, j) \notin B \tag{24}$$

To illustrate how the previous sets of constraints works, consider the example presented in Fig. 4 for a match $(i, j) \notin B$, that is, when only one exchanger is permitted. The hot side of heat exchanger spans intervals 3 through 8 of stream i, the transfer of heat to cold stream j not shown. Since only one exchanger is allowed for this match, variables $Y^{z,H}_{ijm}$ are defined as binary, while $K^{z,H}_{ijm}$ and $\hat{K}^{z,H}_{ijm}$ are continuous. The values of these variables for this example are given in table in the right of Fig. 4. These numbers are consistent with the set of constraints (15)–(19) and (20)–(24) and are uniquely defined by them.

From the figure one can see that whenever $Y^{z,H}_{ijm} = 0$, then it follows that $K^{z,H}_{ijm} = 0$, $\hat{K}^{z,H}_{ijm} = 0$ (constraints (17) and (22)). In turn, at any interval where $Y^{z,H}_{ijm-1} = 1$, constraint (18) becomes trivial and thus, $K^{z,H}_{ijm}$ is forced to zero. Indeed, when $Y^{z,H}_{ijm} = 1$, constraint (16) forces $K^{z,H}_{ijm}$ to zero and when $Y^{z,H}_{ijm} = 0$, $K^{z,H}_{ijm}$ is forced to zero by means of constraint (17). Similarly, when $Y^{z,H}_{ijm+1} = 1$ then $\hat{K}^{z,H}_{ijm}$ is forced to be zero. Now, when a heat exchanger begins (interval 3 in this example), the conditions $Y^{z,H}_{ijm-1} = 0$ and $Y^{z,H}_{ijm} = 1$ are satisfied and thus, $K^{z,H}_{ijm}$ is set to one by means of constraints (16) and (18). Likewise, when a heat exchanger ends (interval 8 in this example), the conditions $Y^{z,H}_{ijm} = 1$ and $Y^{z,H}_{ijm+1} = 0$ force $\hat{K}^{z,H}_{ijm}$ to be one.

Now consider the possibility of allowing two heat exchangers between the same pair of streams, as shown in Fig. 5. In this case, there are two heat exchangers between the shown hot stream and a certain cold stream. For convenience of presentation, the exchangers are placed in series for the hot stream without any other unit in between, but the model is not limited to this situation. The corresponding constraints are:



m	Y_{ijm}	K_{ijm}	$\hat{K}_{ijm}$
1	0	0	0
2	0	0	0
3	**1**	**1**	0
4	**1**	0	0
5	**1**	0	0
6	**2**	**1**	**1**
7	**1**	0	0
8	**1**	0	**1**
9	0	0	0
10	0	0	0

Fig. 5. Heat exchanger definition when $(i, j) \in B$.

Heat exchanger existence on hot streams—$(i, j) \in B$:

$$Y_{ijm}^{z,H} = \sum_{\substack{l \in M_i^z \\ l \le m \\ j \in P_{il}^H}} K_{ijl}^{z,H} - \sum_{\substack{l \in M_i^z \\ l \le m-1 \\ j \in P_{il}^H}} \hat{K}_{ijl}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z; j \in C^z; j \in P_{im}^H; (i, j) \in B \tag{25}$$

Since more than one exchanger is allowed, that is, $(i, j) \in B$, constraint (25) is used for defining heat exchangers existence. In addition, variables $K_{ijm}^{z,H}$ and $\hat{K}_{ijm}^{z,H}$ are declared as binary, while $Y_{ijm}^{z,H}$ are declared continuous. The table on Fig. 5 shows the values of these variables.

Since the binary variables $K_{ijm}^{z,H}$ and $\hat{K}_{ijm}^{z,H}$ are set to one whenever a heat exchanger begins or ends, respectively, then constraint (25) sets $Y_{ijm}^{z,H} \ge 1$ for all intervals m between the beginning and end of a heat exchanger. When a heat exchanger between the same pair of stream ends and another one begins in the same interval (interval 6 in this example) then $Y_{ijm}^{z,H}$ is equal to two. Otherwise it is equal to one. This explains the choice of declaring Y as continuous variables in this case. One could use Equation (25) for all cases, but the distinction is made to reduce the number of binary variables.

A similar set of equations is used to define the location of a heat exchanger for cold streams. These expressions are presented next without further explanation.

Heat exchanger beginning for cold streams—$(i, j) \notin B$:

$$K_{ijn}^{z,C} \ge Y_{ijn}^{z,C} \quad z \in Z; n \in M^z; n = n_j^0; i \in H^z; j \in C_n^z; i \in P_{jn}^C; (i, j) \notin B \tag{26}$$

$$K_{ijn}^{z,C} \le 2 - Y_{ijn}^{z,C} - Y_{ijn-1}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{27}$$

$$K_{ijn}^{z,C} \le Y_{ijn}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{28}$$

$$K_{ijn}^{z,C} \ge Y_{ijn}^{z,C} - Y_{ijn-1}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{29}$$

$$K_{ijn}^{z,C} \ge 0 \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{30}$$

Heat exchanger end for cold streams—$(i, j) \notin B$:

$$\hat{K}_{ijn}^{z,C} \ge Y_{ijn}^{z,C} \quad z \in Z; n \in M^z; n = n_j^0; i \in H^z; j \in C_n^z; i \in P_{jn}^C; (i, j) \notin B \tag{31}$$

$$\hat{K}_{ijn}^{z,C} \le 2 - Y_{ijn}^{z,C} - Y_{ijn-1}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{32}$$

$$\hat{K}_{ijn}^{z,C} \le Y_{ijn}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{33}$$

$$\hat{K}_{ijn}^{z,C} \ge Y_{ijn}^{z,C} - Y_{ijn-1}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{34}$$

$$\hat{K}_{ijn}^{z,C} \ge 0 \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; (i, j) \notin B \tag{35}$$

Heat exchanger existence on cold streams—$(i, j) \in B$:

$$Y_{ijn}^{z,C} = \sum_{\substack{l \in N_j^z \\ l \le n \\ i \in P_{jl}^C}} K_{ijl}^{z,C} - \sum_{\substack{l \in N_j^z \\ l \le n-1 \\ i \in P_{jl}^C}} \hat{K}_{ijl}^{z,C} \quad z \in Z; n \in M^z; i \in H^z; j \in C_n^z; i \in P_{jn}^C; (i, j) \in B \tag{36}$$

Finally, the number of heat exchanger units between a given pair of streams, E_{ij}^z, is obtained by counting the number of beginnings or endings. Since the number of beginnings and endings ought to be equal, this condition is enforced by equating the number of units to the number of beginnings ((37)–(38)) and number of endings ((39)–(40)).

Number of heat exchangers between hot stream i and cold stream j:

$$E_{ij}^z = \sum_{m \in M_i^z; j \in P_{im}^H} K_{ijm}^{z,H} \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P \tag{37}$$

$$E_{ij}^{z} = \sum_{n \in N_j^z;\, i \in P_{jn}^C} K_{ijn}^{z,C} \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P \tag{38}$$

$$E_{ij}^{z} = \sum_{m \in M_i^z;\, j \in P_{im}^H} \hat{K}_{ijm}^{z,H} \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P \tag{39}$$

$$E_{ij}^{z} = \sum_{n \in N_j^z;\, i \in P_{jn}^C} \hat{K}_{ijn}^{z,C} \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P \tag{40}$$

$$E_{ij}^{z} \le 1 \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P; (i, j) \notin B \tag{41}$$

$$E_{ij}^{z} \le E_{ij}^{z,\max} \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P; (i, j) \in B \tag{42}$$

We limit the number of exchangers to be one in the case where one exchanger is allowed, and to a maximum $E_{ij}^{z,\max}$ otherwise.

2.4. Heat transfer consistency

When multiple heat exchangers are allowed between streams *i* and *j*, a new set of constraints has to be added in order to individually account for the heat load of each unit. Since the heat load of a heat exchanger is accounted separately for the hot and cold streams, these equations will ensure the equality of these two values for every exchanger. We illustrate and address the situation depicted in Fig. 6, where the hot stream exchanges heat with a cold stream in two exchangers in a way such that the end of the first exchanger and the beginning of the second exchanger takes place in the same interval. Specifically, for this example, the interval in question is $m = 6$. This interval is such that part of the cumulative heat $\hat{q}_{ij6}^{z,H}$ is sent the interval 5 of the cold stream, and another part to interval 7.

In order to be able to determine the heat loads of each individual exchanger, the model needs to distinguish explicitly which portion of $\hat{q}_{ij6}^{z,H}$ is transferred to interval 5 and which is transferred to interval 7. For this purpose, a new variable $\tilde{q}_{ijm}^{z,H}$ is introduced, which measures the amount of heat that is transferred to the next heat exchanger in the sequence, assuring that calculating the heat load for each exchanger using the hot stream yields the same value than using the cold stream. This new variable is also used later in flow rate consistency constraints and in area calculations for each heat exchanger. In addition, another new variable, $X_{im,jn}^{z}$, is introduced in order to determine the ending interval of each heat exchanger in the sequence for match (i, j). This variable is set to zero whenever *m* and *n* are cold-end intervals, taking positive values in all other cases. We illustrate these equations using Fig. 6.

The heat transfer consistency constraints are presented next.

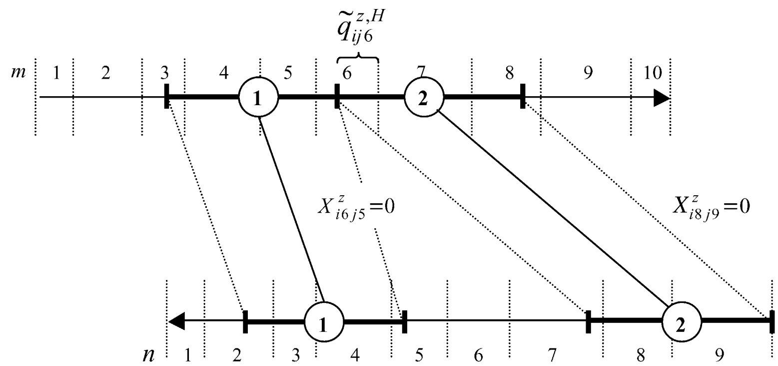
Heat transfer consistency for multiple heat exchangers between the same pair of streams:

$$\sum_{\substack{l \in M_i^z \\ l \le m}} \hat{q}_{ijl}^{z,H} - \tilde{q}_{ijn}^{z,H} \le \sum_{\substack{l \in N_j^z \\ l \le n}} \hat{q}_{ijl}^{z,C} - \tilde{q}_{ijm}^{z,C} + 4X_{im,jn}^{z} \,\mathrm{Max}\left\{ \sum_{\substack{l \in M_i^z \\ l \le m \\ j \in P_{il}^H}} \Delta H_{il}^{z,H};\ \sum_{\substack{l \in M_i^z \\ l \le n \\ i \in P_{jl}^C}} \Delta H_{jl}^{z,C} \right\}$$

$$z \in Z; m, n \in M^z; T_n^L \le T_m^U; (i, j) \in B; i \in H_m^z;\ j \in C_n^z; i \in P_{jn}^C; j \in P_{im}^H \tag{43}$$

$$\sum_{\substack{l \in M_i^z \\ l \le m}} \hat{q}_{ijl}^{z,H} - \tilde{q}_{ijn}^{z,H} \ge \sum_{\substack{l \in N_j^z \\ l \le n}} \hat{q}_{ijl}^{z,C} - \tilde{q}_{ijm}^{z,C} + 4X_{im,jn}^{z} \,\mathrm{Max}\left\{ \sum_{\substack{l \in M_i^z \\ l \le m \\ j \in P_{il}^H}} \Delta H_{il}^{z,H};\ \sum_{\substack{l \in M_i^z \\ l \le n \\ i \in P_{jl}^C}} \Delta H_{jl}^{z,C} \right\}$$

$$z \in Z; m, n \in M^z; T_n^L \le T_m^U; (i, j) \in B; i \in H_m^z;\ j \in C_n^z; i \in P_{jn}^C; j \in P_{im}^H \tag{44}$$



$Load \; = \hat{q}^{z,H}_{ij3} + \hat{q}^{z,H}_{ij4} + \hat{q}^{z,H}_{ij5} + \hat{q}^{z,H}_{ij6} - \tilde{q}^{z,H}_{ij6}$

$Unit\ 1 \; = \hat{q}^{z,C}_{ij2} + \hat{q}^{z,C}_{ij3} + \hat{q}^{z,C}_{ij4} + \hat{q}^{z,C}_{ij5}$

$Load \; = \hat{q}^{z,H}_{ij3} + \hat{q}^{z,H}_{ij4} + \hat{q}^{z,H}_{ij5} + \hat{q}^{z,H}_{ij6} + \hat{q}^{z,H}_{ij7} + \hat{q}^{z,H}_{ij8}$

$Units\ 1{+}2 \; = \hat{q}^{z,C}_{ij2} + \hat{q}^{z,C}_{ij3} + \hat{q}^{z,C}_{ij4} + \hat{q}^{z,C}_{ij5} + \hat{q}^{z,C}_{ij7} + \hat{q}^{z,C}_{ij8} + \hat{q}^{z,C}_{ij9}$

m	Y_m	K_m	$\hat{K}_m$	$\tilde{q}_m$
1	0	0	0	0
2	0	0	0	0
3	**1**	**1**	0	0
4	**1**	0	0	0
5	**1**	0	0	0
6	**2**	**1**	**1**	**≥0**
7	**1**	0	0	0
8	**1**	0	**1**	0
9	0	0	0	0
10	0	0	0	0

n	Y_n	K_n	$\hat{K}_n$	$\tilde{q}_m$
1	0	0	0	0
2	**1**	**1**	0	0
3	**1**	0	0	0
4	**1**	0	0	0
5	**1**	0	**1**	0
6	0	0	0	0
7	**1**	**1**	0	0
8	**1**	0	0	0
9	**1**	0	**1**	0

$X^{z}_{im,\,jn}$

m/n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	2	2	2	2	1.25	2.25	2.25	2.25	1.5
2	2	2	2	2	1.25	2.25	2.25	2.25	1.5
3	2	2	2	2	1.25	2.25	2.25	2.25	1.5
4	2	2	2	2	1.25	2.25	2.25	2.25	1.5
5			2	2	1.25	2.25	2.25	2.25	1.5
6			0.75	0.75	**0**	1	1	1	0.25
7				1.75	1	2	2	2	1.25
8						0.75	0.75	0.75	**0**
9							1.75	1.75	1
10								1.75	1

Fig. 6. Heat transfer consistency example when $(i, j) \in B$.

$$X^{z}_{im,\,jn} = 2 - \hat{K}^{z,H}_{ijm} - \hat{K}^{z,C}_{ijn} + \frac{1}{4} \sum_{\substack{l \in N^{z}_{j} \\ l \le n}} \hat{K}^{z,C}_{ijl} - \frac{1}{4} \sum_{\substack{l \in M^{z}_{i} \\ l \le m}} \hat{K}^{z,H}_{ijl}$$

$$z \in Z; m, n \in M^{z}; T^{L}_{n} \le T^{U}_{m}; (i,\ j) \in B; i \in H^{z}_{m};\ \ j \in C^{z}_{n}; i \in P^{C}_{jn}; j \in P^{H}_{im} \tag{45}$$

$$\sum_{\substack{l \in M^{z}_{i} \\ l \le m \\ j \in P^{H}_{il}}} \hat{K}^{z,H}_{ijl} - \sum_{\substack{l \in N^{z}_{j} \\ l \le n \\ i \in P^{C}_{jl}}} \hat{K}^{z,C}_{ijl} \ge 0 \quad z \in Z; m, n \in M^{z}; T^{L}_{n} < T^{U}_{m}; T^{L}_{n} \ge T^{L}_{m} (i,\ j) \in B; i \in H^{z}_{m}; j \in C^{z}_{n}; i \in P^{C}_{jn}; j \in P^{H}_{im} \tag{46}$$

$$\sum_{\substack{l\in M_i^z\\ l\le m\\ j\in P_{il}^H}} \left(K_{ijl}^{z,H}-\hat{K}_{ijl}^{z,H}\right)\le 1 \quad z\in Z; m\in M^z; (i,j)\in B; i\in H_m^z; j\in P_{im}^H \tag{47}$$

$$\sum_{\substack{l\in N_j^z\\ l\le n\\ i\in P_{jl}^H}} \left(K_{ijl}^{z,H}-\hat{K}_{ijl}^{z,C}\right)\le 1 \quad z\in Z; m\in M^z; (i,j)\in B; i\in H_m^z; j\in P_{im}^H \tag{48}$$

$$\tilde{q}_{ijm}^{z,H}\le \hat{q}_{ijm}^{z,H} \quad z\in Z; m\in M^z; (i,j)\in B; i\in H_m^z; j\in P_{im}^H \tag{49}$$

$$\tilde{q}_{ijm}^{z,H}\le K_{ijm}^{z,H}\,\Delta H_{im}^{z,H} \quad z\in Z; m\in M^z; (i,j)\in B; i\in H_m^z; j\in P_{im}^H \tag{50}$$

$$\tilde{q}_{ijm}^{z,H}\le \hat{K}_{ijm}^{z,H}\,\Delta H_{im}^{z,H} \quad z\in Z; m\in M^z; (i,j)\in B; i\in H_m^z; j\in P_{im}^H \tag{51}$$

$$\tilde{q}_{ijm}^{z,H}\ge 0 \quad z\in Z; m\in M^z; (i,j)\in B; i\in H_m^z; j\in P_{im}^H \tag{52}$$

$$\tilde{q}_{ijn}^{z,C}\le \hat{q}_{ijn}^{z,C} \quad z\in Z; n\in M^z; (i,j)\in B; j\in C_n^z; i\in P_{jn}^C \tag{53}$$

$$\tilde{q}_{ijn}^{z,C}\le K_{ijn}^{z,C}\,\Delta H_{in}^{z,C} \quad z\in Z; n\in M^z; (i,j)\in B; j\in C_n^z; i\in P_{jn}^C \tag{54}$$

$$\tilde{q}_{ijn}^{z,C}\le \hat{K}_{ijn}^{z,C}\,\Delta H_{in}^{z,C} \quad z\in Z; n\in M^z; (i,j)\in B; j\in C_n^z; i\in P_{jn}^C \tag{55}$$

$$\tilde{q}_{ijn}^{z,C}\ge 0 \quad z\in Z; n\in M^z; (i,j)\in B; j\in C_n^z; i\in P_{jn}^C \tag{56}$$

The set of constraints (43)–(45) imposes the condition that the heat load of each heat exchanger has to be equal no matter if it is calculated using the hot stream or the cold stream heat transfer. This constraint is the core of the heat transfer consistency, while the rest are subsidiary to allow the proper calculation of auxiliary variables. In this context, whenever $X_{im,jn}^z$ is equal zero, then constraints (43) and (44) are equivalent to the following equality, which states the heat balance consistency:

$$\sum_{\substack{l\in M_i^z\\ l\le m}} \hat{q}_{ijl}^{z,H}-\tilde{q}_{ijn}^{z,H} = \sum_{\substack{l\in N_j^z\\ l\le n}} \hat{q}_{ijl}^{z,C}-\tilde{q}_{ijm}^{z,C}$$

Looking at Fig. 6, notice that $X_{im,jn}^z$ is zero only for $(m, n) = (6, 5)$ and $(m, n) = (8, 9)$. For $(m, n) = (6, 5)$, constraints (43) and (44) reduce to:

$$\hat{q}_{ij3}^{z,H}+\hat{q}_{ij4}^{z,H}+\hat{q}_{ij5}^{z,H}+\hat{q}_{ij6}^{z,H}-\tilde{q}_{ij6}^{z,H}=\hat{q}_{ij2}^{z,C}+\hat{q}_{ij3}^{z,C}+\hat{q}_{ij4}^{z,C}+\hat{q}_{ij5}^{z,C}-\tilde{q}_{ij5}^{z,C}$$

But since (54) forces $\tilde{q}_{ij5}^{z,C}$ to be zero because a heat exchanger does not start at $n = 5$, then:

$$\hat{q}_{ij3}^{z,H}+\hat{q}_{ij4}^{z,H}+\hat{q}_{ij5}^{z,H}+\hat{q}_{ij6}^{z,H}-\tilde{q}_{ij6}^{z,H}=\hat{q}_{ij2}^{z,C}+\hat{q}_{ij3}^{z,C}+\hat{q}_{ij4}^{z,C}+\hat{q}_{ij5}^{z,C}$$

The above result states that the load of the first heat exchanger, calculated for the hot stream (left-hand side) is equal to the same load calculated for the cold stream. When constraints (43) and (44) are applied to the second case in which $X_{im,jn}^z$ is zero, that is (m, n)=(8, 9), they render the cumulative load of both exchangers. Therefore, given that the load of the first exchanger had been made consistent previously, the load of the second heat exchanger is consistent too.

In turn, constraints (49)–(56) limit $\tilde{q}$ allowing it to be different from zero only at intervals when a heat exchanger ends and a new one begins. For all other intervals, this variable is set to zero.

Constraints (47) and (48) are integer cuts that enforce the condition that a new heat exchanger can only start once the previous one has ended. Notice that these constraints allow contiguous heat exchangers, where the ending of a preceding heat exchanger takes place in the same interval as the beginning of a subsequent unit.

Finally, constraint (46) is another integer cut enforcing the condition that at the cold-end of a heat exchanger the temperature of the hot stream has to be greater than the temperature of the cold stream. Thus, constraint (46) prevents temperature difference infeasibilities in the cold-end of a heat exchanger. Fig. 7 illustrates how this constraint works.

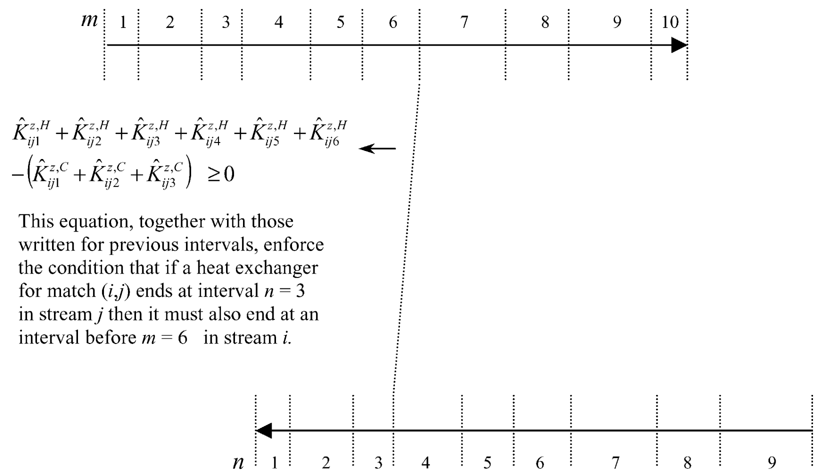


Fig. 7. Integer cut for heat exchanger end when $(i, j) \in B$.

2.5. Flow rate consistency within heat exchangers

The next set of equations represents the consistency of flow rate within a heat exchanger, which state the condition that flow rate passing through a heat exchanger is constant. None of the MILP formulations described previously in the literature (Furman & Sahinidis, 2002) consider using this condition explicitly, resulting in heat loads that are not uniformly distributed along the temperature intervals. Thus, area calculations cannot be precisely carried out in those models. The reason for this is that most of these previous formulations were meant to be targeting devices, not necessarily design devices. As we shall see next, in our new formulation, the flow rate consistency allows for more precise area values (only exact if the number of temperature intervals is sufficiently large).

Fig. 8 illustrates a heat exchanger spanning intervals 3 through 8 for hot stream i exchanging heat with cold stream j. In this case, intervals 3 and 8 are referred to as “extreme” intervals, while the remaining intervals (4 through 7) are denoted as “exchanger-internal”.

Consider for a moment the case where only one exchanger is allowed for match (i, j). Then, notice that for the exchanger-internal intervals, the flow rate can be consistently determined as the ratio between the cumulative heat transfer, the heat capacity and the interval temperature range. On the other hand, for the extreme intervals (3 and 8) heat is being exchanged with some other cold stream(s) using the remaining portion of the interval. Thus, for extreme intervals the mentioned ratio always underestimates the real flow rates because the real temperature range for heat exchange is smaller than the interval range. Therefore, only inequality constraints can be written for extreme intervals (see Fig. 8)

To distinguish whether an interval is “exchanger-internal” or not, a new variable, α, is defined as one, with the exception being the first internal interval, which also receives the value of zero. This is needed to properly pose the flow rate consistency equations. This variable is declared as continuous but the following set of constraints forces it to take a value of one if the interval is exchanger-internal and zero otherwise.

Definition of exchanger-internal intervals for hot streams:

$$\alpha_{ijm}^{z,H} \leq 1 - K_{ijm}^{z,H} - K_{ijm-1}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in P_{im}^H \cap P_{im-1}^H; i \in S^H; j \in C^Z \tag{57}$$

$$\alpha_{ijm}^{z,H} \leq 1 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijm-1}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in P_{im}^H \cap P_{im-1}^H; i \in S^H; j \in C^Z \tag{58}$$

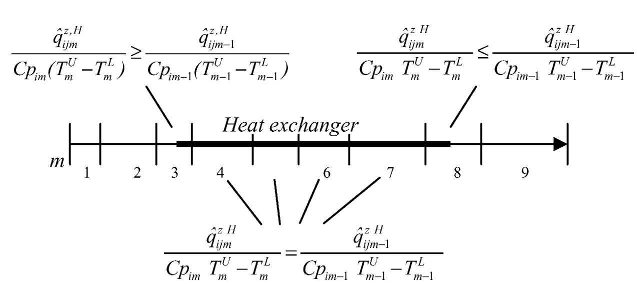


Fig. 8. Flow rate consistency equations.

$$\alpha_{ijm}^{z,H} \geq Y_{ijm}^{z,H} - K_{ijm}^{z,H} - K_{ijm-1}^{z,H} - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijm-1}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in P_{im}^H \cap P_{im-1}^H; i \in S^H; j \in C^Z \tag{59}$$

$$\alpha_{ijm}^{z,H} \geq 0 \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in P_{im}^H \cap P_{im-1}^H; i \in S^H; j \in C^Z \tag{60}$$

When the interval is exchanger-internal, (except the first internal) $K_{ijm}^{z,H}$, $K_{ijm-1}^{z,H}$, $\hat{K}_{ijm}^{z,H}$, $\hat{K}_{ijm-1}^{z,H}$ are zero and $Y_{ijm}^{z,H} = 1$; hence, $\alpha_{ijm}^{z,H}$ is forced to be equal to one by constraints (57), or (58) and (59). Conversely, for extreme intervals, at least one of $K_{ijm}^{z,H}$, $K_{ijm-1}^{z,H}$, $\hat{K}_{ijm}^{z,H}$, $\hat{K}_{ijm-1}^{z,H}$ will be equal to one and since $Y_{ijm}^{z,H} = 1$, the above equations force $\alpha_{ijm}^{z,H}$ to zero. In the case of the first internal interval (as in interval 4 of Fig. 8) $K_{ijm-1}^{z,H} = 1$, then $\alpha_{ijm}^{z,H}$ will also be set to zero.

Using this definition for α, the flow rate consistency constraints for hot streams are given next. When splits are not allowed for the particular stream a simplified set of equations is used to reduce the number of equations needed. In addition, when more than one heat exchanger is allowed between the same pair of streams a different set of constraints is used that takes into account the variables $\tilde{q}$ described before. We present the flow rate consistency equations next. We first cover non-splits and only one exchanger.

Flow rate consistency for hot streams in exchanger-internal intervals—$i \in S^H$, $(i, j) \in B$:

$$\frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Cp}_{im}(T_m^U - T_m^L)} \leq \frac{\hat{q}_{ijm-1}^{z,H}}{\mathrm{Cp}_{im-1}(T_{m-1}^U - T_{m-1}^L)} + (1-\alpha_{ijm}^{z,H})F_i \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; i \in S^H; j \in C^z; j \in P_{im}^H \cap P_{im-1}^H \tag{61}$$

$$\frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Cp}_{im}(T_m^U - T_m^L)} \geq \frac{\hat{q}_{ijm-1}^{z,H}}{\mathrm{Cp}_{im-1}(T_{m-1}^U - T_{m-1}^L)} - (1-\alpha_{ijm}^{z,H})F_i \quad z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; i \in S^H; j \in C^z; j \in P_{im}^H \cap P_{im-1}^H \tag{62}$$

When $\alpha = 1$, that is, the interval is exchanger-internal, then the two constraints render equality of flows between intervals m and $m-1$. Conversely, when $\alpha = 0$, which corresponds to end intervals, the first internal interval or to intervals in which heat transfer between streams i and j does not take place, then the above inequalities become trivial. Since the inequalities involve two consecutive intervals, this explains why α is set to zero for the interval next to the beginning of the exchanger. For extreme intervals, the following set of inequalities is needed to assure flow rate consistency.

Flow rate consistency for hot streams in extreme intervals—$i \in S^H$, $(i, j) \notin B$:

$$\frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Cp}_{im}(T_m^U - T_m^L)} \geq \frac{\hat{q}_{ijm-1}^{z,H}}{\mathrm{Cp}_{im-1}(T_{m-1}^U - T_{m-1}^L)} - (1 + \hat{K}_{ijm-1}^{z,H} + \hat{K}_{ijm}^{z,H} - K_{ijm-1}^{z,H})F_i$$
$$z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in P_{im}^H \cap P_{im-1}^H; i \in S^H; j \in C^Z; (i, j) \notin B \tag{63}$$

$$\frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Cp}_{im}(T_m^U - T_m^L)} \leq \frac{\hat{q}_{ijm-1}^{z,H}}{\mathrm{Cp}_{im-1}(T_{m-1}^U - T_{m-1}^L)} + (1 + K_{ijm-1}^{z,H} + K_{ijm}^{z,H} - \hat{K}_{ijm}^{z,H})F_i$$
$$z \in Z; m \in M^z; i \in H_m^z \cap H_{m-1}^z; j \in P_{im}^H \cap P_{im-1}^H; i \in S^H; j \in C^Z; (i, j) \notin B \tag{64}$$

Constraint (63) applies to the beginning (hot-end) of a heat exchanger while inequality (64) applies to the end of a heat exchanger (cold-end). Notice that the last term of the right-hand side of (63) vanishes when the end of a heat exchanger occurs, rendering the desired inequality. The equation considers three cases. First, when $K_{ijm-1}^{z,H} = 1$ and $\hat{K}_{ijm-1}^{z,H} = \hat{K}_{ijm}^{z,H} = 0$, i.e. there are more internal intervals, then (63) renders the desired equality of flows. A similar thing happens at the end of the exchanger by Equation (64). On the other hand, for an internal interval, both constraints become trivial. Notice these constraints are always trivial for heat exchangers spanning less than three intervals, since there are not internal intervals in those cases.

In the case where $(i, j) \in B$; this is, more than one heat exchanger is allowed between streams i and j, an equivalent set of constraints is defined which considers the possibility of having the beginning and the end of two different heat exchangers in the same interval.

Flow rate consistency for hot streams in extreme intervals—$i \in S^H$, $(i, j) \in B$:

$$\frac{\hat{q}^{z,H}_{ijm}}{\mathrm{Cp}_{im}(T^U_m - T^L_m)} \geq \frac{\hat{q}^{z,H}_{ijm-1}}{\mathrm{Cp}_{im-1}(T^U_{m-1} - T^L_{m-1})} - \left(1 + \hat{K}^{z,H}_{ijm-1} + \hat{K}^{z,H}_{ijm} - K^{z,H}_{ijm-1}\right) F_i$$

$$z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m-1}; j \in P^H_{im} \cap P^H_{im-1}; i \in S^H; j \in C^Z; (i, j) \in B \quad (65)$$

$$\frac{\hat{q}^{z,H}_{ijm}}{\mathrm{Cp}_{im}(T^U_m - T^L_m)} \geq \frac{\tilde{q}^{z,H}_{ijm-1}}{\mathrm{Cp}_{im-1}(T^U_{m-1} - T^L_{m-1})} - \left(2 + \hat{K}^{z,H}_{ijm} - K^{z,H}_{ijm-1} - Y^{z,H}_{ijm-1}\right) F_i$$

$$z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m-1}; j \in P^H_{im} \cap P^H_{im-1}; i \in S^H; j \in C^Z; (i, j) \in B \quad (66)$$

$$\frac{\hat{q}^{z,H}_{ijm} - \tilde{q}^{z,H}_{ijm}}{\mathrm{Cp}_{im}(T^U_m - T^L_m)} \leq \frac{\tilde{q}^{z,H}_{ijm-1}}{\mathrm{Cp}_{im-1}(T^U_{m-1} - T^L_{m-1})} + \left(2 + K^{z,H}_{ijm-1} - \hat{K}^{z,H}_{ijm} - Y^{z,H}_{ijm}\right) F_i$$

$$z \in Z; m \in M^z; i \in H^z_m \cap H^z_{m-1}; j \in P^H_{im} \cap P^H_{im-1}; i \in S^H; j \in C^Z; (i, j) \in B \quad (67)$$

In this case, constraint (65) is valid when a heat exchanger begins (hot-end) at interval $m-1$ while constraint (66) is used for the beginning interval when there is other exchanger ending in the interval as well. In turn, constraint (67) is valid at the end of a heat exchanger, regardless of whether one is following or not.

Although the sets of constraints (63)–(67) are valid in a general case, they can be simplified if splits are not allowed for the corresponding hot stream. This is because when splits are not allowed for hot stream i, then the flow rate calculated for exchanger-internal intervals has to be equal to the actual stream flow rate. This condition is enforced using the following constraints.

Flow rate consistency for hot streams—$i \notin S^H$:

$$\hat{q}^{z,H}_{ijm} \geq \left(Y^{z,H}_{ijm} - K^{z,H}_{ijm} - \hat{K}^{z,H}_{ijm}\right) \Delta H^{z,H}_{im} \quad z \in Z; m \in M^z; i \in H^z_{m-1} \cap H^z_m \cap H^z_{m+1}; i \notin S^H; j \in C^z; j \in P^H_{im-1} \cap P^H_{im} \cap P^H_{im+1} \quad (68)$$

Constraint (68) enforces the heat flow to be equal to the enthalpy change for any internal interval. This is equivalent to set the flow rate passing through the heat exchanger equal to the stream actual flow rate. In the case where $(i, j) \in B$ the same concept is applied but now two constraints are required since the Y variables may take values greater than one.

Flow rate consistency constraints for cold streams are given next without further explanation.

Definition of exchanger-internal intervals for cold streams—$j \in S^C$:

$$\alpha^{z,C}_{ijn} \leq 1 - K^{z,C}_{ijn} - K^{z,C}_{ijn-1} \quad z \in Z; n \in M^z; j \in C^z_n \cap C^z_{n-1}; j \in S^C; i \in H^z; i \in P^C_{jn} \cap P^C_{jn-1} \quad (69)$$

$$\alpha^{z,C}_{ijn} \leq 1 - \hat{K}^{z,C}_{ijn} - \hat{K}^{z,C}_{ijn-1} \quad z \in Z; n \in M^z; j \in C^z_n \cap C^z_{n-1}; j \in S^C; i \in H^z; i \in P^C_{jn} \cap P^C_{jn-1} \quad (70)$$

$$\alpha^{z,C}_{ijn} \geq Y^{z,C}_{ijn} - K^{z,C}_{ijn} - K^{z,C}_{ijn-1} - \hat{K}^{z,C}_{ijn} - \hat{K}^{z,C}_{ijn-1} \quad z \in Z; n \in M^z; j \in C^z_n \cap C^z_{n-1}; j \in S^C; i \in H^z; i \in P^C_{jn} \cap P^C_{jn-1} \quad (71)$$

$$\alpha^{z,C}_{ijn} \geq 0 \quad z \in Z; n \in M^z; j \in C^z_n \cap C^z_{n-1}; j \in S^C; i \in H^z; i \in P^C_{jn} \cap P^C_{jn-1} \quad (72)$$

Flow rate consistency for cold streams in exchanger-internal intervals—$j \in S^C$, $(i, j) \notin B$:

$$\frac{\hat{q}^{z,C}_{ijn}}{\mathrm{Cp}_{jn}(T^U_n - T^L_n)} \leq \frac{\hat{q}^{z,C}_{ijn-1}}{\mathrm{Cp}_{jn-1}(T^U_{n-1} - T^L_{n-1})} + \left(1 + \alpha^{z,C}_{ijn}\right) F_j \quad z \in Z; n \in M^z; j \in S^C\, j \in C^z_n \cap C^z_{n-1}; i \in H^z; i \in P^C_{jn} \cap P^C_{jn-1} \quad (73)$$

$$\frac{\hat{q}^{z,C}_{ijn}}{\mathrm{Cp}_{jn}(T^U_n - T^L_n)} \geq \frac{\hat{q}^{z,C}_{ijn-1}}{\mathrm{Cp}_{jn-1}(T^U_{n-1} - T^L_{n-1})} - \left(1 - \alpha^{z,C}_{ijn}\right) F_j \quad z \in Z; n \in M^z; j \in S^C; j \in C^z_n \cap C^z_{n-1}; i \in H^z; i \in P^C_{jn} \cap P^C_{jn-1} \quad (74)$$

Flow rate consistency for cold streams in extremes intervals—$j \in S^C$, $(i, j) \notin B$:

$$\frac{\hat{q}_{ijn}^{z,C}}{\mathrm{Cp}_{jn}(T_n^U - T_n^L)} \geq \frac{\hat{q}_{ijn-1}^{z,C}}{\mathrm{Cp}_{jn}(T_n^U - T_n^L)} - \left(1 + \hat{K}_{ijn-1}^{z,C} + \hat{K}_{ijn}^{z,C} - K_{ijn-1}^{z,C}\right) F_i$$

$$z \in Z; n \in M^z; (i, j) \notin B; j \in S^C; j \in C_n^z \cap C_{n-1}^z; i \in H^z; i \in P_{jn}^C \cap P_{jn-1}^C \tag{75}$$

$$\frac{\hat{q}_{ijn}^{z,C}}{\mathrm{Cp}_{jn}(T_n^U - T_n^L)} \leq \frac{\hat{q}_{ijn-1}^{z,C}}{\mathrm{Cp}_{in-1}(T_{n-1}^U - T_{n-1}^L)} + \left(1 + K_{ijn-1}^{z,C} + K_{ijn}^{z,C} - \hat{K}_{ijn}^{z,C}\right) F_i$$

$$z \in Z; n \in M^z; (i, j) \notin B; j \in S^C; j \in C_n^z \cap C_{n-1}^z; i \in H^z; i \in P_{jn}^C \cap P_{jn-1}^C \tag{76}$$

Flow rate consistency for cold streams in extreme intervals—$j \in S^C$, $(i, j) \in B$:

$$\frac{\hat{q}_{ijn}^{z,C}}{\mathrm{Cp}_{jn}(T_n^U - T_n^L)} \geq \frac{\hat{q}_{ijn-1}^{z,C}}{\mathrm{Cp}_{jn-1}(T_{n-1}^U - T_{n-1}^L)} + \left(1 + \hat{K}_{ijn-1}^{z,C} + \hat{K}_{ijn}^{z,C} - K_{ijn-1}^{z,C}\right) F_j$$

$$z \in Z; n \in M^z; (i, j) \in B; j \in S^C; j \in C_n^z \cap C_{n-1}^z; i \in H^z; i \in P_{jn}^C \cap P_{jn-1}^C \tag{77}$$

$$\frac{\hat{q}_{ijn}^{z,C}}{\mathrm{Cp}_{jn}(T_n^U - T_n^L)} \geq \frac{\tilde{q}_{ijn-1}^{z,C}}{\mathrm{Cp}_{jn-1}(T_{n-1}^U - T_{n-1}^L)} - \left(2 + \hat{K}_{ijn}^{z,C} - K_{ijn-1}^{z,C} - Y_{ijn-1}^{z,C}\right) F_j$$

$$z \in Z; n \in M^z; (i, j) \in B; j \in S^C; j \in C_n^z \cap C_{n-1}^z; i \in H^z; i \in P_{jn}^C \cap P_{jn-1}^C \tag{78}$$

$$\frac{\hat{q}_{ijn}^{z,C} - \tilde{q}_{ijn}^{z,C}}{\mathrm{Cp}_{jn}(T_n^U - T_n^L)} \leq \frac{\hat{q}_{ijn-1}^{z,C}}{\mathrm{Cp}_{jn-1}(T_{n-1}^U - T_{n-1}^L)} + \left(2 + K_{ijn-1}^{z,C} - K_{ijn-1}^{z,C} - Y_{ijn}^{z,C}\right) F_j$$

$$z \in Z; n \in M^z; (i, j) \in B; j \in S^C; j \in C_n^z \cap C_{n-1}^z; i \in H^z; i \in P_{jn}^C \cap P_{jn-1}^C \tag{79}$$

Flow rate consistency for cold streams—$j \notin S^C$:

$$\hat{q}_{ijn}^{z,C} \geq \left(Y_{ijn}^{z,C} - K_{ijn}^{z,C} - \hat{K}_{ijn}^{z,C}\right) \Delta H_{jn}^{z,C}$$

$$z \in Z; n \in M^z; j \in C_{n-1}^z \cap C_n^z \cap C_{n+1}^z; j \notin S^C(i, j) \notin B; i \in H^z; i \in P_{jn-1}^C \cap P_{jn}^C \cap P_{jn+1}^C \tag{80}$$

2.6. Temperature difference enforcing

Equations enforcing a temperature difference $\Delta T_{\min}$ are written at the beginning and end of a heat exchanger, which are important in order to assure network feasibility. This condition is already guaranteed for exchanger-internal intervals because of the shift of scales. However, for extreme intervals, it is necessary to include additional constraints.

Let us first examine the case where splits are not allowed, as shown in Fig. 9. In this case, the minimum temperature difference at each end of a heat exchanger is enforced by means of the depicted inequalities. Notice that these constraints are linear only because when no splits are allowed, the flow rate passing through a heat exchanger is the total stream flow rate, which is a

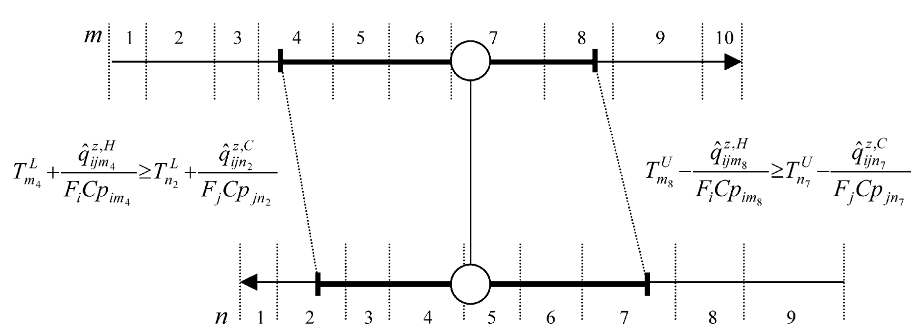


Fig. 9. Temperature difference assurance when splits are not allowed.

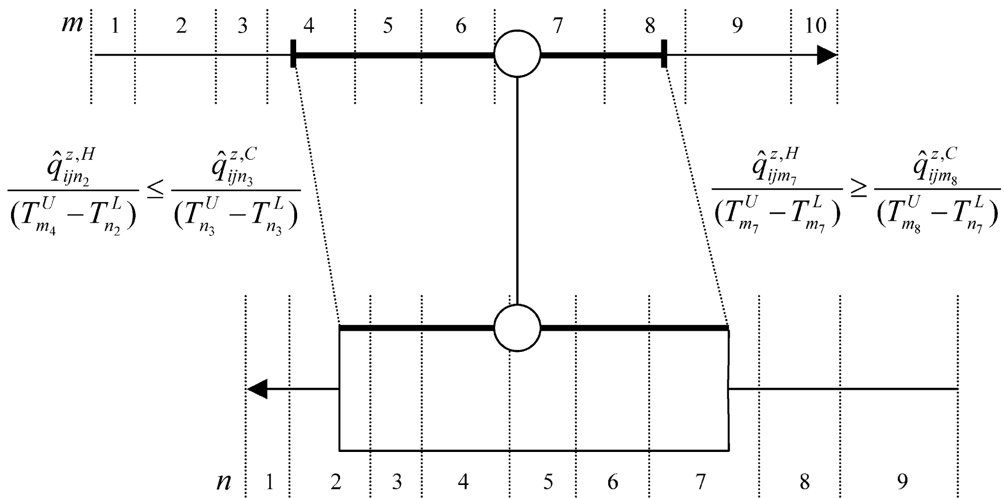


Fig. 10. Temperature difference assurance when splits are allowed.

parameter of the model. The constraints enforcing a minimum approach temperature for the case where splits are not allowed are presented next.

Temperature feasibility constraints—$i \notin S^H, j \notin S^C$:

$$T_m^L + \frac{\hat{q}_{ijm}^{z,H}}{F_i \mathrm{Cp}_{im}} \ge T_n^L + \frac{\hat{q}_{ijn}^{z,C}}{F_j \mathrm{Cp}_{jn}} - (2 - K_{ijm}^{z,H} - K_{ijn}^{z,C}) T_n^U$$

$$z \in Z; m, n \in M^z; T_n^L \le T_m^U; T_n^U \ge T_m^L; i \in H_m^z; j \in C_n^z; i \notin S^H; j \notin S^C; i \in P_{jn}^C; j \in P_{im}^H \tag{81}$$

$$T_m^U - \frac{\hat{q}_{ijm}^{z,H}}{F_i \mathrm{Cp}_{im}} \ge T_n^U - \frac{\hat{q}_{ijn}^{z,C}}{F_j \mathrm{Cp}_{jn}} - (2 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C}) T_n^U$$

$$z \in Z; m, n \in M^z; T_n^L \le T_m^U; T_n^U \ge T_m^L; i \in H_m^z; j \in C_n^z; i \notin S^H; j \notin S^C; i \in P_{jn}^C; j \in P_{im}^H \tag{82}$$

Note that the last term of the right-hand side of (81) and (82) only vanishes when a heat exchanger starts or ends at overlapping temperature intervals for the hot and cold streams, rendering the proper inequalities. In all other cases, the inequalities are trivial.

Now consider the case where stream splits are permitted, as illustrated in Fig. 10. Two different sets of constraints are defined for cases where only one or multiple heat exchangers are permitted. The equations for cases where $(i, j) \notin B$ are presented first, and then the correspondent to cases where $(i, j) \in B$ are given.

Temperature feasibility constraints—$i \in S^H, j \in S^C, (i, j) \notin B$:

$$\hat{K}_{ijn}^{z,C} \le 2 - K_{ijm}^{z,H} - K_{ijn}^{z,C}$$

$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \tag{83}$$

$$\frac{\hat{q}_{ijn}^{z,C}}{T_m^U - T_n^L} \le \frac{\hat{q}_{ijn+1}^{z,C}}{T_{n+1}^U - T_{n+1}^L} \frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn+1}} + (2 - K_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C}) \frac{\Delta H_{jn}^{z,C}}{T_m^U - T_n^L}$$

$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \tag{84}$$

$$\frac{\hat{q}_{ijm}^{z,C}}{\mathrm{Min}\left\{T_m^U; T_n^U\right\} - T_m^L} \ge \frac{\hat{q}_{ijm+1}^{z,C}}{T_{m+1}^U - T_{m+1}^L} \frac{\mathrm{Cp}_{im}}{\mathrm{Cp}_{im+1}} - (2 - K_{ijm}^{z,H} - K_{ijn}^{z,C}) \frac{\Delta H_{im+1}^{z,H}}{T_{m+1}^U - T_{n+1}^L}$$

$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \tag{85}$$

$$K_{ijm}^{z,H} \leq 2 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \tag{86}$$

$$\frac{\hat{q}_{ijm}^{z,H}}{T_m^U - T_n^L} \leq \frac{\hat{q}_{ijm-1}^{z,H}}{T_{m-1}^U - T_{m-1}^L} \frac{\mathrm{Cp}_{im}}{\mathrm{Cp}_{im-1}} + (2 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C}) \frac{\Delta H_{im}^{z,H}}{T_m^U - T_n^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \tag{87}$$

$$\frac{\hat{q}_{ijn}^{z,C}}{T_n^U - \mathrm{Max}\left\{T_m^L; T_n^L\right\}} \geq \frac{\hat{q}_{ijn-1}^{z,C}}{T_{n-1}^U - T_{n-1}^L} \frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn-1}} + (2 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C}) \frac{\Delta H_{jn-1}^{z,H}}{T_{n-1}^U - T_{n-1}^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \tag{88}$$

Note that (83)–(85) and (86)–(88) are written only for overlapping pairs of intervals where $T_n^L < T_m^U; T_n^U > T_m^L$. When the temperature intervals overlap but do not represent the beginning (hot-end) of a heat exchanger, constraints (83)–(85) become trivial since the last terms of the right-hand sides are positive. Consider first the case, where m and n correspond to the hot-end of a heat exchanger at overlapping intervals. In such case, $K_{ijm}^{z,H} = K_{ijn}^{z,C} = 1$ and thus the last term of the right-hand side of Equations (83)–(85) vanishes, giving the following set of constraints:

$$\hat{K}_{ijn}^{z,C} \leq 0; \frac{\hat{q}_{ijn}^{z,C}}{T_m^U - T_n^L} \leq \frac{\hat{q}_{ijn+1}^{z,C}}{T_{n+1}^U - T_{n+1}^L} \frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn+1}}; \frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Min}\left\{T_m^U; T_n^U\right\} - T_m^L} \geq \frac{\hat{q}_{ijm+1}^{z,H}}{T_{m+1}^U - T_{m+1}^L} \frac{\mathrm{Cp}_{im}}{\mathrm{Cp}_{im+1}}$$

The first of these inequalities forbids that the cold-end of the heat exchanger for the cold stream be located at the same temperature interval than the hot-end. In turn, the second inequality forces the hot-end temperature of the cold stream to be lower than the upper temperature limit of the hot-end interval for the hot stream. Indeed,

$$\frac{\hat{q}_{ijn}^{z,C}}{T_m^U - T_n^L} \leq \frac{\hat{q}_{ijn+1}^{z,C}}{T_{n+1}^U - T_{n+1}^L} \frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn+1}} \Rightarrow T_n^L \leq T_m^U - \frac{\hat{q}_{ijn}^{z,C}}{\hat{q}_{ijn+1}^{z,C}} \frac{\mathrm{Cp}_{jn+1}}{\mathrm{Cp}_{jn}} (T_{n+1}^U - T_{n+1}^L) = T_m^U - \frac{\hat{q}_{ijn}^{z,C}}{\mathrm{Cp}_{jn}}$$

Finally, the third inequality forces the hot-end temperature of the hot stream to be larger than $\mathrm{Min}\{T_m^U; T_n^U\}$. Fig. 11 illustrates the two possible cases for $\mathrm{Min}\{T_m^U; T_n^U\}$. Indeed,

$$\frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Min}\{T_m^U; T_n^U\} - T_m^L} \geq \frac{\hat{q}_{ijm+1}^{z,H}}{T_{m+1}^U - T_{m+1}^L} \frac{\mathrm{Cp}_{im}}{\mathrm{Cp}_{im+1}} \Rightarrow T_m^L + \frac{\hat{q}_{ijm}^{z,H}}{\hat{q}_{ijm+1}^{z,H}} \frac{\mathrm{Cp}_{im+1}}{\mathrm{Cp}_{im}} (T_{m+1}^U - T_{m+1}^L) \geq \mathrm{Min}\{T_m^U; T_n^U\}$$

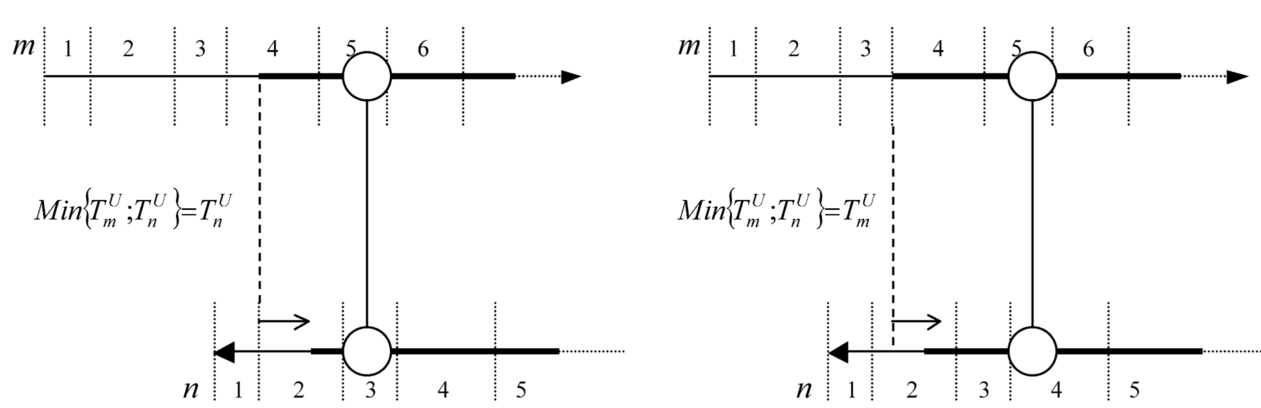


Fig. 11. Temperature difference assurance at the hot-end of an exchanger—$i \in S^H$, $j \in S^C$, $(i, j) \notin B$.

or, equivalently: $T_m^L + \frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Cp}_{im}} = \mathrm{Min}\{T_m^U; T_n^U\}$. The temperature differences at the exchanger cold-end (constraints (86)–(88)) is enforced in an equivalent manner.

We now present the equations corresponding to more than one exchanger

Temperature feasibility constraints—$i \in S^H, j \in S^C, (i,j) \in B$:

$$\hat{K}_{ijn}^{z,C} \leq 1 + Y_{ijn}^{z,C} - K_{ijm}^{z,H} - K_{ijn}^{z,C}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \quad (89)$$

$$\frac{\hat{q}_{ijn}^{z,C}}{T_m^U - T_n^L} \leq \frac{\hat{q}_{ijn+1}^{z,C}}{T_{n+1}^U - T_{n+1}^L}\frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn+1}} + (1 + Y_{ijn}^{z,C} - K_{ijm}^{z,H} - K_{ijn}^{z,C})\frac{\Delta H_{jn}^{z,C}}{T_m^U - T_n^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \quad (90)$$

$$\frac{\tilde{q}_{ijn}^{z,C}}{T_m^U - T_n^L} \leq \frac{\hat{q}_{ijn+1}^{z,C}}{T_{n+1}^U - T_{n+1}^L}\frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn+1}} + (2 - K_{ijm}^{z,H} - K_{ijn}^{z,C})\frac{\Delta H_{jn}^{z,C}}{T_m^U - T_n^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m+1}^z; j \in C_n^z \cap C_{n+1}^z; i \in P_{jn}^C \cap P_{jn+1}^C; j \in P_{im}^H \cap P_{im+1}^H \quad (91)$$

$$\frac{\hat{q}_{ijm}^{z,H}}{\mathrm{Min}\{T_m^U; T_n^U\} - T_m^L} \geq \frac{\hat{q}_{ijm+1}^{z,H}}{T_{m+1}^U - T_{m+1}^L}\frac{\mathrm{Cp}_{im}}{\mathrm{Cp}_{im+1}} - (2 - K_{ijm}^{z,H} - K_{ijn}^{z,C})\frac{\Delta H_{im+1}^{z,H}}{T_{m+1}^U - T_{m+1}^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m-1}^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; j \in P_{im}^H \cap P_{im-1}^H \quad (92)$$

$$K_{ijm}^{z,H} \leq 1 + Y_{ijm}^{z,H} - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m-1}^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; j \in P_{im}^H \cap P_{im-1}^H \quad (93)$$

$$\frac{\hat{q}_{ijm}^{z,H} - \tilde{q}_{ijm}^{z,H}}{T_m^U - T_n^L} \leq \frac{\hat{q}_{ijm-1}^{z,H}}{T_{m-1}^U - T_{m-1}^L}\frac{\mathrm{Cp}_{im}}{\mathrm{Cp}_{im-1}} + (2 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C})\frac{\Delta H_{im}^{z,H}}{T_m^U - T_n^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m-1}^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; j \in P_{im}^H \cap P_{im-1}^H \quad (94)$$

$$\frac{\hat{q}_{ijm}^{z,H} - \tilde{q}_{ijm}^{z,H}}{T_n^U - \mathrm{Max}\{T_m^L, T_n^L\}} \leq \frac{\hat{q}_{ijn-1}^{z,C}}{T_{n-1}^U - T_{n-1}^L}\frac{\mathrm{Cp}_{jn}}{\mathrm{Cp}_{jn-1}} - (2 - \hat{K}_{ijm}^{z,H} - \hat{K}_{ijn}^{z,C})\frac{\Delta H_{jn-1}^{z,C}}{T_{n-1}^U - T_{n-1}^L}$$
$$z \in Z; m, n \in M^z; i \in S^H; j \in S^C; T_n^L < T_m^U; T_n^U > T_m^L; i \in H_m^z \cap H_{m-1}^z; j \in C_n^z \cap C_{n-1}^z; i \in P_{jn}^C \cap P_{jn-1}^C; j \in P_{im}^H \cap P_{im-1}^H \quad (95)$$

2.7. Heat exchanger area calculation

This section describes a series of constraints that are used to estimate the heat exchanger area required to perform the heat transfer of any stream match. When only one heat exchanger is permitted between streams *i* and *j*, the required heat transfer area is simply calculated as:

$$A_{ij}^{z} = \sum_{m \in M_i^z} \sum_{\substack{n \in N_j^z; T_n^L < T_m^U \\ j \in P_{im}^H; i \in P_{jn}^C}} \left[\frac{q_{im,jn}^{z}(h_{im} + h_{jn})}{\Delta T_{mn}^{\mathrm{ML}} h_{im} h_{jn}} \right] \quad z \in Z; i \in H^z; j \in C^z; (i,j) \in P \tag{96}$$

In the case where multiple heat exchangers between streams i and j are permitted, the previous equation gives the total required area for the match (i, j), that is, the summation of the areas of all exchangers placed for the match. However, the designer wants to know the area required for each individual exchanger. This is facilitated by the following set of constraints.

$$\hat{A}_{ij}^{z,k} \leq \sum_{\substack{l \in M_i^z \\ l \leq m}} \sum_{\substack{n \in N_j^z \\ T_n^L < T_m^U \\ j \in P_{im}^H \\ i \in P_{jn}^C}} \left[\frac{(q_{il,jn}^{z} - \breve{q}_{il,jn}^{z}) \cdot (h_{il} + h_{jn})}{\Delta T_{\ln}^{\mathrm{ML}} \cdot h_{il} \cdot h_{jn}} \right] - \sum_{h=1}^{k-1} A_{ij}^{z,h} + A_{ij\,\max}^{z}(2 - \hat{K}_{ijm}^{z,H} - G_{ijm}^{z,k})$$

$$z \in Z; m \in M^z; i \in H_m^z;\ j \in C^z; j \in P_{im}^H; (i,j) \in B; k = 1, \ldots, k_{\max} - 1 \tag{97}$$

$$\hat{A}_{ij}^{z,k} \geq \sum_{\substack{l \in M_i^z \\ l < m}} \sum_{\substack{n \in N_j^z \\ T_n^L < T_m^U \\ j \in P_{im}^H \\ i \in P_{jn}^C}} \left[\frac{(q_{il,jn}^{z} - \breve{q}_{il,jn}^{z}) \cdot (h_{il} + h_{jn})}{\Delta T_{\ln}^{\mathrm{ML}} \cdot h_{il} \cdot h_{jn}} \right] - \sum_{h=1}^{k-1} A_{ij}^{z,h} - A_{ij\,\max}^{z}(2 - \hat{K}_{ijm}^{z,H} - G_{ijm}^{z,k})$$

$$z \in Z; m \in M^z; i \in H_m^z;\ j \in C^z; j \in P_{im}^H; (i,j) \in B; k = 1, \ldots, k_{\max} - 1 \tag{98}$$

$$\hat{A}_{ij}^{z,k} \geq A_{ij}^{z} - \sum_{h=1}^{k-1} \hat{A}_{ij}^{z,h} \quad z \in Z;\ m \in M^z;\ i \in H_m^z;\ \ j \in C^z; j \in P_{im}^H; (i,j) \in B; k = 1, \ldots, k_{\max} - 1 \tag{99}$$

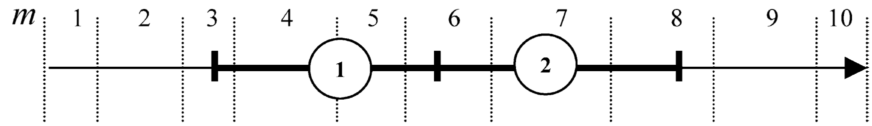
$$\sum_{h=1}^{k_{\max}} h \cdot G_{ijm}^{z,h} = \sum_{\substack{l \in M_i^z; l \leq m \\ j \in P_{im}^H}} K_{ijl}^{z,H} + 1 - Y_{ijm}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z;\ j \in C^z; j \in P_{im}^H; (i,j) \in B; k = 1, \ldots, k_{\max} - 1 \tag{100}$$

$$\sum_{\substack{n \in N_j^z; T_n^L < T_m^U \\ j \in P_{im}^H; i \in P_{jn}^C}} \breve{q}_{im,jn}^{z} = \tilde{q}_{ijm}^{z,H} \quad z \in Z; m \in M^z; i \in H_m^z;\ j \in C^z; j \in P_{im}^H; (i,j) \in B; k = 1, \ldots, k_{\max} - 1 \tag{101}$$

$$\breve{q}_{im,jn}^{z} \leq q_{im,jn}^{z} \quad z \in Z; m \in M^z; i \in H_m^z;\ j \in C^z; j \in P_{im}^H; (i,j) \in B; k = 1, \ldots, k_{\max} - 1 \tag{102}$$

Using these constraints, the area of the k-th heat exchanger is calculated by subtracting the area of the previous $k-1$ exchangers to the total accumulated area until the end of the k-th exchanger. To do that, a parameter defining the maximum number of heat exchangers allowed per match ($k_{\max}$) is required. Additionally, new binary variables ($G_{ijm}^{z,k}$) are introduced to determine which exchanger is present at a certain temperature interval. In this context, whenever $(2 - \hat{K}_{ijm}^{z,H} - G_{ijm}^{z,k})$ equals zero, constraints (97) and (98) allow the determination of the required area for the k-th heat exchanger. On the other hand, when $(2 - \hat{K}_{ijm}^{z,H} - G_{ijm}^{z,k}) > 0$, these two constraints become trivial. An example showing the values of this term is presented in Fig. 12.

For a maximum of two exchangers, the equations determining the values of $G_{ijm}^{z,1}$ and $G_{ijm}^{z,2}$ are $G_{ijm}^{z,1} + 2G_{ijm}^{z,2} = \sum_{l \in M_i^z; l \leq m; j \in P_{im}^H} K_{ijl}^{z,H} + 1 - Y_{ijm}^{z,H}$. Thus, for the first five intervals, we have $G_{ijm}^{z,1} + 2G_{ijm}^{z,2} = 1$, which forces $G_{ijm}^{z,2} = 0$ and



G^1	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
G^2	0	0	0	0	0	**1**	**1**	**1**	**1**	**1**
$2 - \hat{K} - G^1$	1	1	1	1	1	**0**	2	2	2	2
$2 - \hat{K} - G^2$	2	2	2	2	2	1	1	**0**	1	1

Fig. 12. Area computation when $(i, j) \in B$.

$G_{ijm}^{z,1} = 1$. In interval 6, we have $G_{ijm}^{z,1} + 2G_{ijm}^{z,2} = 2$, which forces $G_{ijm}^{z,1} = 0$ and $G_{ijm}^{z,2} = 1$. As a result, $(2 - \hat{K}_{ijm}^{z,H} - G_{ijm}^{z,1}) = 0$ for $m = 6$ and $(2 - \hat{K}_{ijm}^{z,H} - G_{ijm}^{z,1}) = 0$ for $m = 8$. Thus, defining the proper interval limits to define the areas of the exchangers.

In order to account appropriately for the heat load of each exchanger, new variables $\breve{q}_{im,jn}^{z}$ are defined in equation (101) in relation to the variables previously introduced to preserve heat transfer consistency ($\tilde{q}_{ijm}^{z,H}$). Finally, notice that the set of constraints (97)–(102) is only written for hot stream intervals. An equivalent set could be defined for the cold streams that would give the same results, since only one set suffices for the purposes of area computation. One chooses the set with smallest number of intervals.

2.8. Number of shells

In practice, only a limited amount of area can be packed in a single exchanger shell. The number of shells is defined through an integer variable (U_{ij}^z). Thus, the following constraints need to be added to count the number of shells necessary to perform the resulting heat transfer. The first equation applies when $(i, j) \notin B$, while constraint (103) is used when $(i, j) \in B$.

Maximum shell area:

$$A_{ij}^z \leq A_{ij\,\max}^z U_{ij}^z \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P; (i, j) \notin B \tag{103}$$

$$\hat{A}_{ij}^{z,k} \leq A_{ij\,\max}^z \hat{U}_{ij}^{z,k} \quad z \in Z; i \in H^z; j \in C^z; (i, j) \in P; (i, j) \notin B \tag{104}$$

Notice that since a fixed-charge cost is associated with the number of shells in the objective function, the optimal solution will naturally tend to drive U_{ij}^z to a minimum.

2.9. Objective function

Finally, the objective for the formulation is to minimize the annualized total cost, which includes both operative and capital cost. The expression for the total annual cost is:

$$\text{Min cost} = \sum_{z} \sum_{i \in HU^z} \sum_{\substack{j \in C^z \\ (i,j) \in P}} c_i^H F_i^H \Delta T_i + \sum_{z} \sum_{j \in \mathrm{CU}^z} \sum_{\substack{i \in H^z \\ (i,j) \in P}} c_j^C F_j^C \Delta T_j + \left[\sum_{z} \sum_{i \in H^z} \sum_{\substack{j \in C^z \\ (i,j) \in P}} (c_{ij}^F U_{ij}^z + c_{ij}^A A_{ij}^z) \right]_{(i,j) \notin B}$$
$$+ \left[\sum_{k} \left\{ \sum_{z} \sum_{i \in H^z} \sum_{\substack{j \in C^z \\ (i,j) \in P}} (c_{ij}^F \hat{U}_{ij}^{z,k} + c_{ij}^A \hat{A}_{ij}^{z,k}) \right\} \right]_{(i,j) \in B} \tag{105}$$

We use a linear expression to approximate the cost of a heat exchanger, as opposed to the traditional concave expression. This is a simplifying assumption that does not affect the validity of the results as we show below.

3. Examples

We now present a series of results obtained with the proposed formulation for a variety of examples found in the literature. We simply show in each subsection a set of tables containing the data and the results with a figure showing the final network. The model was coded in GAMS and solved with CPLEX 7.0 using default options. The optimization times that are reported correspond to runs performed in a PC with a 1.0 GHz processor and 2.0 Gb of ram memory.

3.1. Problem 4S1

This problem consisting of two cold and two hot process streams, one cooling and one heating utility was reported in Shenoy (1995). The problem was solved using two heat transfer zones, defined by the pinch temperature resulting of enforcing a minimum temperature difference of 20 °C. The resulting network has a minimum number of units and is close to minimum area. We report actual areas. To illustrate the assertion made above that the linear approximation does not introduce large error, we illustrate in differences between the real cost formula and its linear approximations. In the range of interest (values above 20), the differences are smaller than 3%. As errors in the values of heat transfer coefficients can be of the same size, we conclude that this is as far as one will ever be able to go, short of including the detailed exchanger design procedure into the model. This situation is the same for the rest of the examples (Tables 1–4, Figs. 13 and 14).

3.2. Problem 7SP4

This problem, taken from Papoulias and Grossmann (1983), consists of six hot and one cold process streams, one cooling and one heating utility. The problem was solved using two heat transfer zones defined by the pinch temperature corresponding to a $\Delta T_{\min}$ of 20 °C (Tables 5–8, Fig. 15).

Table 1
Cost data for problem 4S1

Cost data	
HE cost formula (K$)	$10 + 0.8A^{0.8}$
Linear formula ($/year)	$5291.9 + 77.79A$
Plant life (Year)	5
ROR (%)	10
Annual factor (Year^{-1})	0.3221

Table 2
Stream data for problem 4S1

Stream	F (tonnes/h)	Cp (kJ/kg-C)	T_{IN} (°C)	T_{OUT} (°C)	H (MJ/h-m^2-C)	Q (MJ/h)
I1	10.0	1.0	175.0	45.0	0.2	1300.0
I2	40.0	1.0	125.0	65.0	0.2	2400.0
I3	605.0	1.0	180.0	179.0	0.2	605.0
J1	20.0	1.0	20.0	155.0	0.2	2700.0
J2	15.0	1.0	40.0	112.0	0.2	1080.0
J3	52.5	1.0	15.0	25.0	0.2	525.0

$\Delta T_{\min} = 20$ °C.

Table 3
Resulting HEN for problem 4S1

HE	Load (MJ/h)	Area (m^2)
1	395.0	99.8
2	105.0	48.4
3	605.0	160.2
4	275.0	77.4
5	525.0	109
6	1700.0	551.4
7	700.0	312.4
		1358.7

Table 4
Model statistics for problem 4S1

Model statistics	
Single variables	381
Discrete variables	61
Single equations	1012
Non-zero elements	3202
Time to reach a feasible solution (s)	1
Time to reach global optimality (s)	2
Optimality gap at first feasible solution (%)	4.96
B and *B* nodes to first feasible solution	42
B and *B* nodes to reach global optimality	112

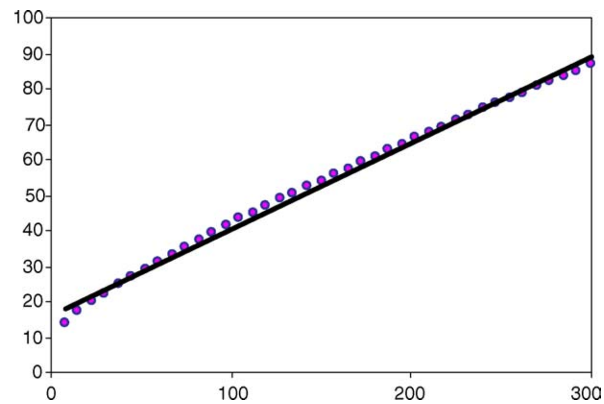


Fig. 13. Comparison between actual and linearized costs.

The resulting network structure is different from the one originally reported by the authors, mainly because they used the classical transshipment model to determine the minimum number of units and then found a network using the resulting matches and heat loads, but without any consideration on the area requirements. Our model not only finds a design with minimum number of units, even though the matches are different, but also with minimum area, thus minimizing the total cost.

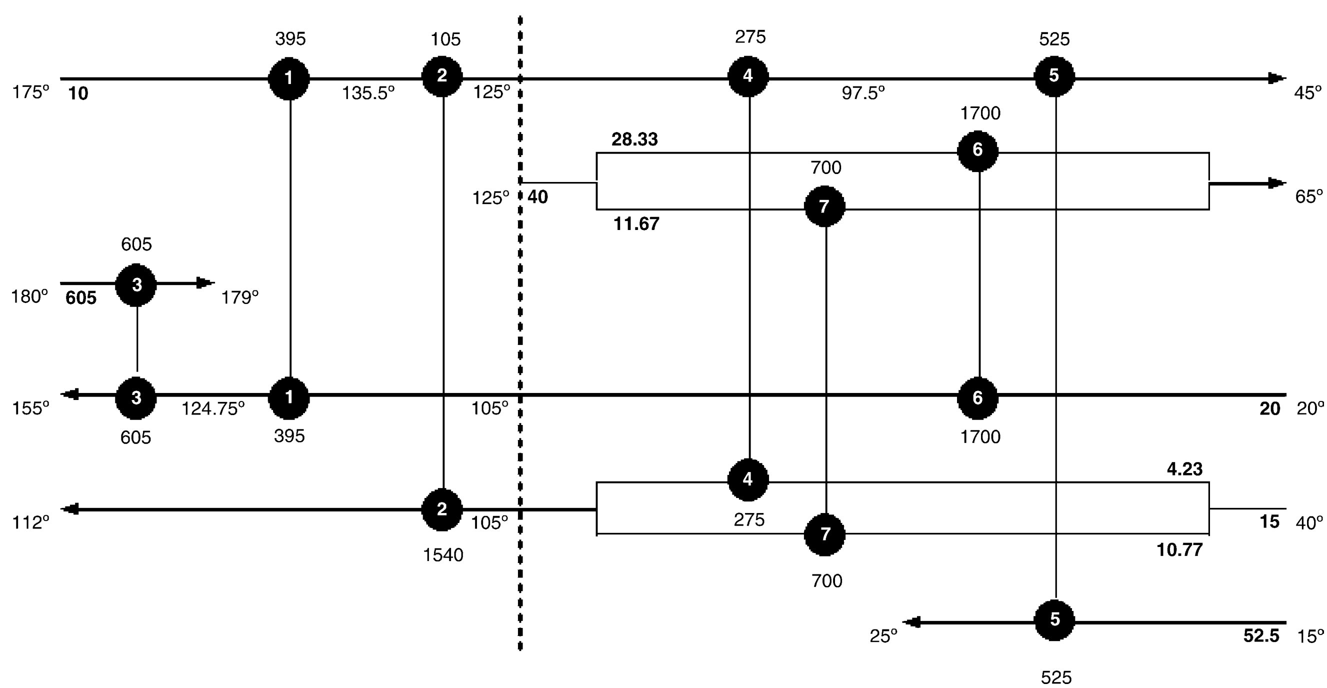


Fig. 14. Final heat exchanger network for problem 4S1.

Table 5
Cost data for problem 7SP4

Cost data	
HE cost formula (K$)	$10 + 0.8A^{0.8}$
Linear formula ($/year)	$5291.9 + 77.79A$
Plant life (Year)	5
ROR (%)	10
Annual factor (Year^{-1})	0.3221

Table 6
Stream data for problem 7SP4

Stream	F (tonnes/h)	Cp (KJ/kg-C)	T_{IN} (°C)	T_{OUT} (°C)	H (MJ/h-m^2-C)	Q (MJ/h)
I1	15.0	1.0	675.0	150.0	0.2	7875.00
I2	11.0	1.0	590.0	450.0	0.2	1540.00
I3	4.5	1.0	540.0	115.0	0.2	1912.50
I4	60.0	1.0	430.0	345.0	0.2	5100.00
I5	12.0	1.0	400.0	100.0	0.2	3600.00
I6	125.0	1.0	300.0	230.0	0.2	8750.00
I7	8390.0	1.0	801.0	800.0	0.2	8390.00
J1	47.0	1.0	60.0	710.0	0.2	30550.00
J2	110.3	1.0	80.0	140.0	0.2	6617.50

$\Delta T_{min} = 20\,°C$.

3.3. Problem 10SP1

This problem consisting of five cold and five hot process streams, one cooling and one heating utility was reported in Cerda (1980), Papoulias and Grossmann (1983). The problem was solved using only one heat transfer zone and a minimum temperature difference of 10 °C (Tables 9–12, Fig. 16).

For the same reasons discussed in example 7SP4, the resulting matches and network structure are different from those originally reported by the authors. Our network features minimum number of units and a total area close to the global minimum.

Table 7
Resulting HEN for problem 7SP4

HE	Load (MJ/h)	Area (m^2)
1	3675.0	676.0
2	1540.0	266.9
3	495.0	135.9
4	8390.0	511.4
5	4200.0	902.4
6	1417.5	409.7
7	5100.0	1027.3
8	3600.0	337.4
9	2132.5	393.1
10	6617.5	427.1
		5087.1

Table 8
Model statistics for problem 7SP4

Model statistics	
Single variables	1106
Discrete variables	147
Single equations	2620
Non-zero elements	9348
Time to reach a feasible solution (s)	10
Time to reach global optimality (s)	60
Optimality gap at first feasible solution (%)	6.9
B and *B* nodes to feasibility	39
B and *B* nodes to reach global optimality	1430

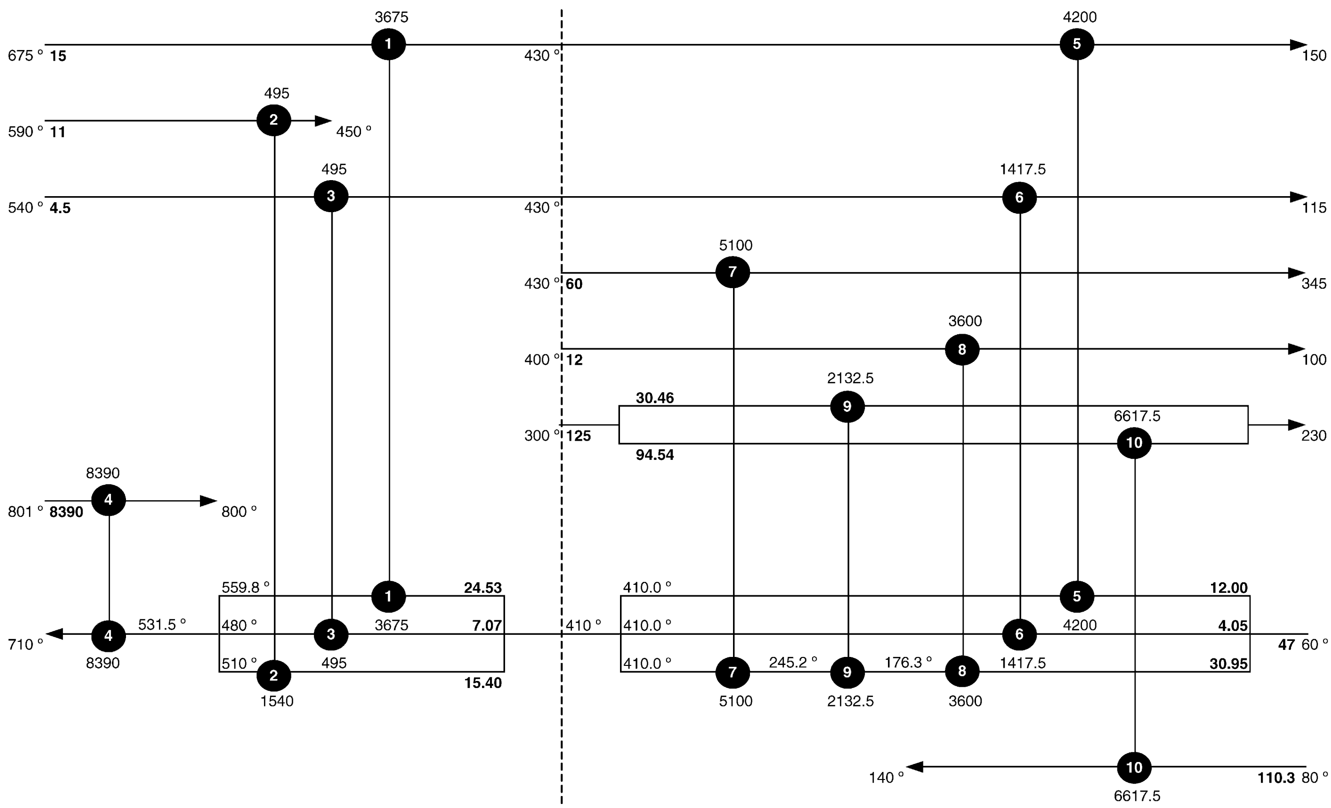


Fig. 15. Final heat exchanger network for problem 7SP4.

3.4. Problem EX1

This problem was taken from Gundersen and Grossmann (1990) and consists of two cold and three hot process streams, one cooling and one heating utility. The problem was solved using two heat transfer zones, defined by the pinch temperature

Table 9
Cost data for problem 10SP1

Cost data	
HE cost formula (K$)	$10 + 0.8A^{0.8}$
Linear formula ($/year)	$5291.9 + 77.79A$
Plant life (Year)	5
ROR (%)	10
Annual factor (Year^{-1})	0.3221

Table 10
Stream data for problem 10SP1

Stream	F (tonnes/h)	Cp (KJ/kg-C)	T_{IN} (°C)	T_{OUT} (°C)	H (MJ/h-m^2-C)	Q (MJ/h)
I1	8.790	1.0	160.0	93.0	0.2	588.93
I2	10.540	1.0	249.0	138.0	0.2	1169.94
I3	14.770	1.0	227.0	66.0	0.2	2377.97
I4	12.560	1.0	271.0	149.0	0.2	1532.32
I5	17.730	1.0	199.0	66.0	0.2	2358.09
J1	7.620	1.0	60.0	160.0	0.2	762.00
J2	6.080	1.0	116.0	222.0	0.2	644.48
J3	8.440	1.0	38.0	221.0	0.2	1544.52
J4	17.280	1.0	82.0	177.0	0.2	1641.60
J5	13.900	1.0	93.0	205.0	0.2	1556.80
J6	42.678	1.0	38.0	82.0	0.2	1877.85

$\Delta T_{min} = 10\,°C$.

Table 11
Resulting HEN for problem 10SP1

HE	Load (MJ/h)	Area (m^2)
1	588.9	109.9
2	644.5	264.0
3	525.5	118.1
4	1641.6	396.4
5	736.4	238.8
6	501.0	84.6
7	1031.3	263.3
8	762.0	146.8
9	454.6	142.2
10	1141.5	306.4
		2070.4

resulting from a minimum temperature difference of 10 °C. In addition, non-isothermal split mixing was permitted in this example. The resulting network structure is similar to that reported by the authors as global optimum, but the split flow rates and outlet temperatures are different due to the ability of our MILP model to approach optimality (the authors originally solved the problem using a non-linear formulation). Consequently, the network obtained with our new methodology has lower total area and cost, proving that the solution reported previously is not optimal (Tables 13–16, Fig. 17).

Table 12
Model statistics for problem 10SP1

Model statistics	
Single variables	1428
Discrete variables	245
Single equations	3711
Non-zero elements	12352
Time to reach a feasible solution (s)	40
Time to reach global optimality (s)	260
Optimality gap at first feasible solution (%)	6.7
B and *B* nodes to first feasible solution	311
B and *B* nodes to reach global optimality	2149

Table 13
Cost data for problem EX1

Cost data	
HE cost formula (K$)	$8.6 + 0.67A^{0.83}$
Linear formula ($/year)	$8153.9 + 61.75A$
Plant life (Year)	5
ROR (%)	10
Annual factor ($Year^{-1}$)	0.3221

Table 14
Stream data for problem EX1

Stream	F (tonnes/h)	Cp (KJ/kg-C)	T_{IN} (°C)	T_{OUT} (°C)	H (MJ/h-m^2-C)	Q (MJ/h)
I1	228.50	1.0	159.0	77.0	0.40	18737.0
I2	20.40	1.0	267.0	88.0	0.30	3651.6
I3	53.80	1.0	343.0	90.0	0.25	13611.4
I4	106.45	1000.0	376.0	375.9	1.00	10645.2
J1	93.30	1.0	26.0	127.0	0.15	9423.3
J2	196.10	1.0	118.0	265.0	0.50	28826.7
J3	559.68	1.0	15.0	30.0	0.60	8395.2

$\Delta T_{min} = 10$ °C.

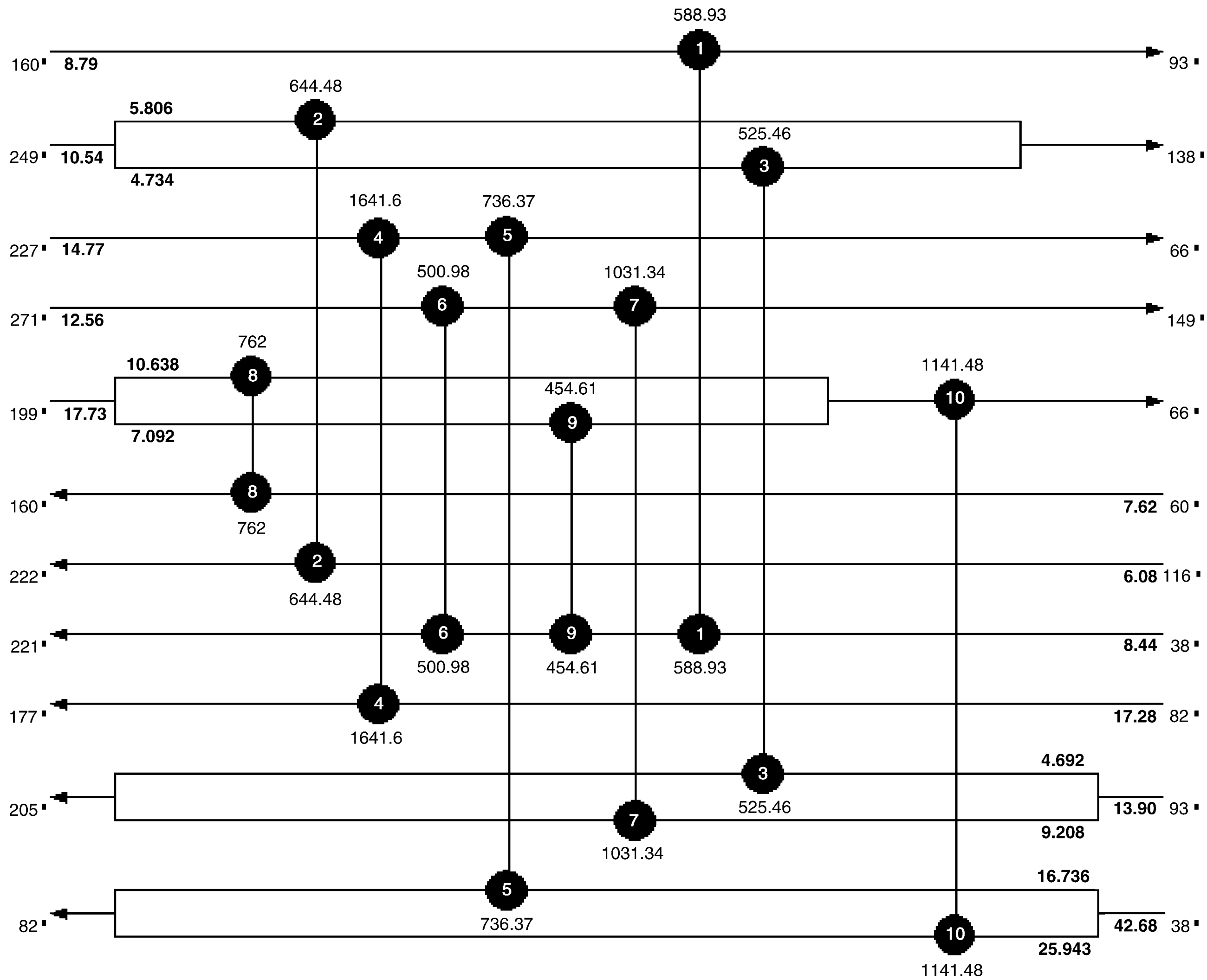


Fig. 16. Final heat exchanger network for problem 10SP1.

3.5. Problem EX2

This problem was taken from Bagajewicz, Rodera, & Savelski (2002) and consists of two cold and three hot process streams and one heating utility. The problem was solved using a single heat transfer zone and a minimum temperature difference of 10 °C. In addition, non-isothermal split mixing was permitted in this example. This example has the interesting property that a network with minimum number of units predicted by the traditional transshipment formulation (Papoulias & Grossmann, 1983) is infeasible (Tables 17–20, Fig. 18).

Table 15
Resulting HEN for problem EX1

HE	Load (MJ/h)	Area (m^2)
1	2203.2	425.6
2	9899.2	1258.6
3	10645.2	234.5
4	4262.7	958.9
5	6079.1	2269.4
6	8395.2	483.6
7	1448.4	407.1
8	3712.2	959.3
		6997.0

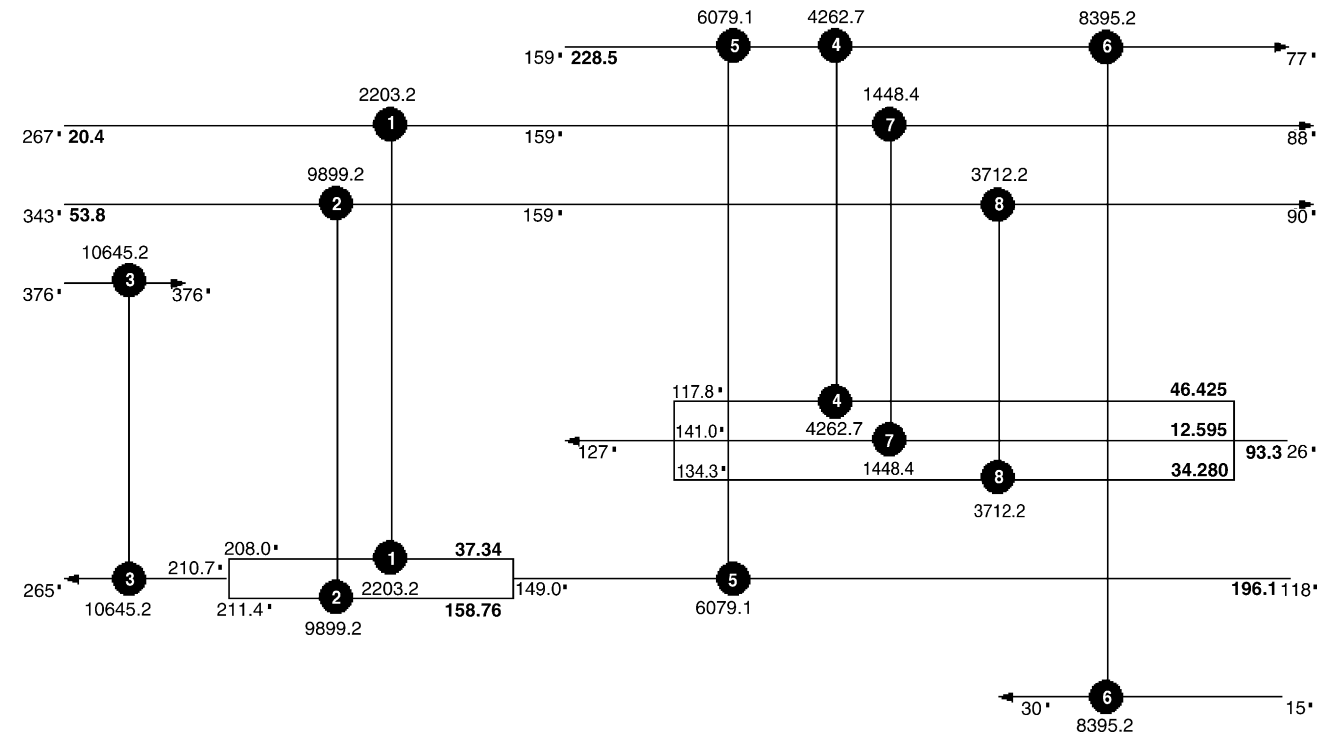


Fig. 17. Final heat exchanger network for problem EX1.

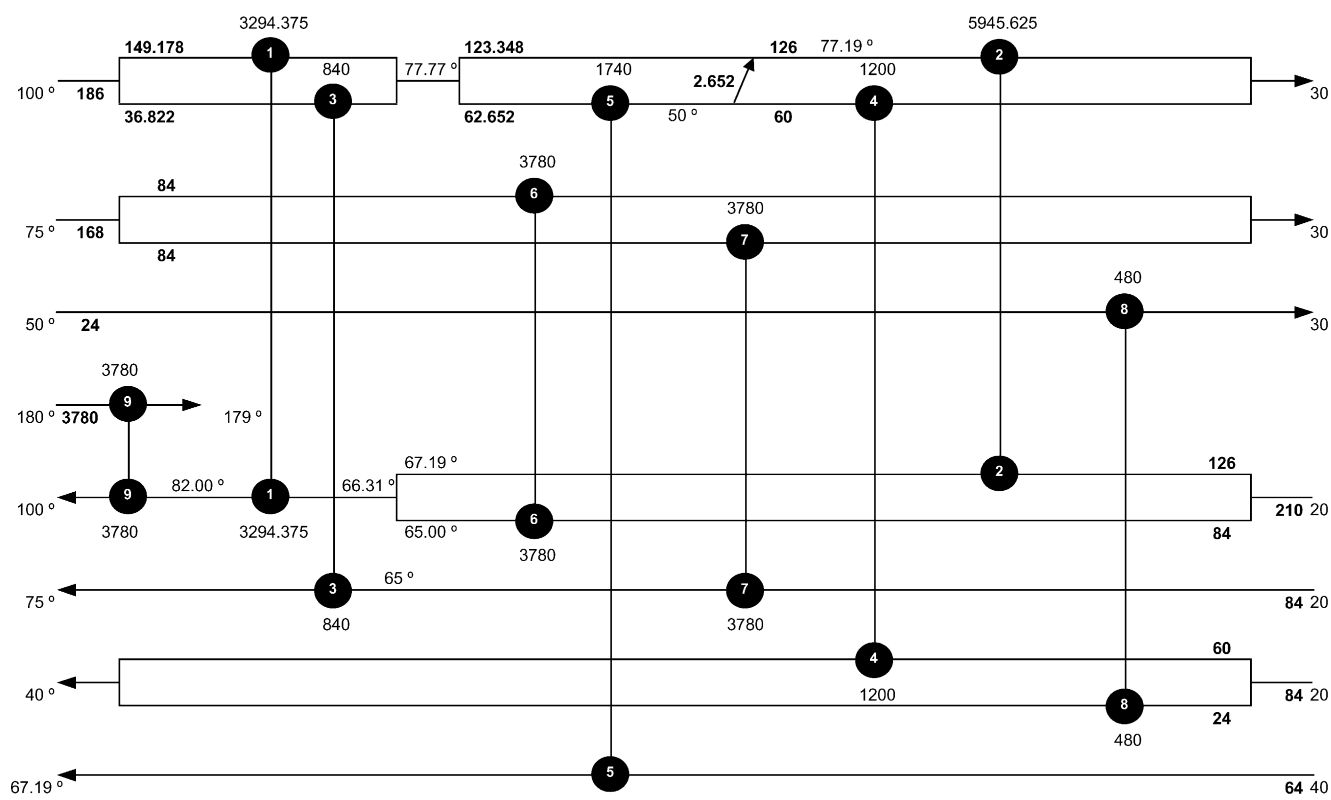


Fig. 18. Final heat exchanger network for problem EX2.

Table 16
Model statistics for problem EX1

Model statistics	
Single variables	1151
Discrete variables	154
Single equations	2704
Non-zero elements	10065
Time to reach feasibility (s)	10
Time to reach global optimality (s)	70
Optimality gap at first feasible solution (%)	1.9
B and *B* nodes to feasibility	124
B and *B* nodes to reach global optimality	1639

Table 17
Cost data for problem EX2

Cost data	
HE cost formula (K$)	$10 + 0.8A^{0.8}$
Linear formula ($/year)	$9498.8 + 58.95A$
Plant life (year)	5
ROR (%)	10
Annual factor (Year^{-1})	0.3221

Table 18
Stream data for problem EX2

Stream	FCp	T_{IN}	T_{OUT}	H	Hot/cold	Q
I1	186.00	100.00	30.00	0.40	1	13020
I2	168.00	75.00	30.00	0.40	1	7560
I3	24.00	50.00	30.00	0.40	1	480
I4	3780.00	180.00	179.00	0.40	1	3780
J1	210.00	20.00	100.00	0.40	0	16800
J2	84.00	20.00	75.00	0.40	0	4620
J3	84.00	20.00	40.00	0.40	0	1680
J4	64.00	40.00	67.19	0.40	0	1740

$\Delta T_{min} = 10\,^{\circ}$C.

Table 19
Resulting HEN for problem EX2

HE	Load (MJ/h)	Area (m^2)
1	3570	1265.5
2	5670	2657.8
3	840.0	228.1
4	1200.0	823.0
5	1740.0	1066.0
6	3780.0	1890.0
7	3780.0	1890.0
8	480.0	240.0
9	3780.0	214.2
		10018

4. Limitations of the model and future work

This is a model based on a transportation/transshipment scheme that requires partitioning streams in intervals. It is therefore easy to predict that the result will depend in some way on the number of intervals each stream is divided and the pattern of such intervals. Conceivably, as the number of intervals is increased, the results will reach some asymptotic pattern of exchange network and corresponding areas. After we obtained the results shown above, we solved the problem for different number of intervals. Table 21 shows the statistics for different number of intervals. The runs render the same network topology with the same loads.

Table 20
Statistics for problem EX2

Model statistics	
Single variables	1197
Discrete variables	180
Single equations	2684
Non-zero elements	13596
Time to reach feasibility (s)	20
Time to reach global optimality (s)	200
Optimality gap at first feasible solution (%)	4.9
B and *B* nodes to feasibility	306
B and *B* nodes to reach global optimality	5324

Table 21
Model statistics for problem 4S1

Number of intervals	26	56	112
Single variables	396	994	2650
Discrete variables	71	139	268
Single equations	1079	2479	10399
Non-zero elements	3175	8359	20719
Time to reach optimality @ (0% gap) (s)	0.4	8.1	165.7
Total area	1374.5	1379.85	1363.97

A second issue of concern is that real costs cannot be incorporated because of the linearity in the approximation of the heat exchanger costs. Thus, costs can be over or underestimated in different portions of the area range. Aside from introducing a truly non-linear (usually concave) objective function, one can overcome this limitation by introducing piece-wise linear functions that can be modeled using special ordered sets (in GAMS). We have not experimented with this, in the believe that the cost equations already carry an inherent uncertainty that makes it focusing on this issue rather fruitless. Rather, we think, the issue should be handled by truly designing under cost uncertainty, an effort that belongs to our future work.

Combinatorial complexity, that is, the increasing number of binary variables needed when the number of intervals is increased, as well as when the problems are larger (with more streams) is of concern. In future work, we will address this limitation.

Another minor limitation, this time related to implementation, is that specific exchanger minimum approximation temperature (EMAT) values cannot be easily imposed, unless the stream temperature partition is done carefully. Indeed, if EMAT values are to be imposed, then one needs to make these partitions, multiple of EMAT, so that specific exchanges between intervals can be excluded from the model. However, EMAT constraints (together with the use of the heat recovery approximation temperature, HRAT) have been proposed to be able to control the trade off between area and energy costs. Given the capabilities of the present model to actually assess automatically this trade off, we see no reason why EMAT constraints are really needed. Nevertheless, they can be incorporated.

5. Conclusions

A newly developed MILP model for grassroots design and retrofit of heat exchanger networks was presented in this article. The use of a special transshipment/transportation structure and the strategic definition of binary variables allow the model to incorporate most of the features that have been identified as shortcomings of previous formulations. The model can handle stream splitting, by-passes, non-isothermal mixing and is capable of counting units, even shells, a capability that allows good cost assessment. The key elements of the model are flow rate consistency constraints that can help handle splitting situations. In addition, especial constraints can “count” heat exchanger shells. The one-step MILP structure of the model constitutes a major conceptual advantage. Additionally, the model easily allows design flexibility if, in fact, the user wants to actively interact during the design stage. This is achieved by fixing, allowing or forbidding topologies using the binary variables that define the heat exchanger network structure. Finally, several examples have been presented that illustrate the power and potential of this new formulation to obtain cost-optimal networks. Some advances can be made to improve this model, as pointed out in Section 4 are left for future work.

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