1 | Kinds of Errors

In discussing errors in individual observations, it is customary to distinguish between systematic errors and chance or random errors.

Systematic errors are errors associated with the particular instruments or technique of measurement being used. Suppose we have a book which is 9 in. high. We measure its height by laying a ruler against it, with one end of the ruler at the top end of the book. If the first inch of the ruler has been previously cut off, then the ruler is likely to tell us that the book is 10 in. long. This is a systematic error. If a thermometer immersed in boiling pure water at normal pressure reads 102°F, it is improperly calibrated. If readings from this thermometer are incorporated into experimental results, a systematic error results. An ammeter which is not properly "zeroed" introduces a systematic error.

Very often, in experimental work, systematic errors are more important than chance errors. They are also, however, much more difficult to deal with. There are no general principles for avoiding systematic errors; only an experimenter whose skill has come through long experience can consistently detect systematic errors and prevent or correct them.

Random errors are produced by a large number of unpredictable and unknown variations in the experimental situation. They can result from small errors in judgment on the part of the observer, such as in estimating tenths of the smallest scale division. Other causes are unpredictable fluctuations in conditions, such as temperature, illumination, line voltage, or any kind of mechanical vibrations of the equipment. It is found empirically that such random errors are frequently distributed according to a simple law. This makes it possible to use statistical methods to deal with random errors. This statistical treatment will form the principal body of the following discussion.

There is a third class, containing what are sometimes called errors but which are not, properly speaking, errors at all. These include mistakes in recording numbers, blunders of reading instruments incorrectly, and mistakes in arithmetic. These types of inaccuracies have no place in a well-done experiment. They can always be eliminated completely by careful work.

The terms accuracy and precision are often used to distinguish between systematic and random errors. If a measurement has small systematic errors, we say that it has high accuracy; if small random errors, we say it has high precision.

2 | Propagation of Errors

Propagation of errors is nothing but a fancy way of describing the obvious fact that if one uses various experimental observations to calculate a result, and if the observations have errors associated with them, then
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\[ \ln V_i = \ln V_0 - \frac{t_i}{RC} \]  \hspace{1cm} (15.25)

Introducing a new variable \( y \), defined by \( y = \ln V_i \), we now have a linear observation equation in \( y \) and \( t_i \). Furthermore, because of the logarithmic scale, it is reasonable to assume that the errors in the \( y \) are of the same variance; so we may proceed with exactly the same methods which led to Eqs. (15.11). If the scale is not logarithmic, the \( y \) will not have the same variance, however.

It is not always possible to reduce an equation to linear form by a simple substitution. In more complicated cases it may be expedient to calculate approximate values of the unknown quantities and then represent the nonlinear equations by linear approximations, using Taylor series expansions.

4. It sometimes happens that we do not know the form of the observation equations or, indeed, whether the observed quantities are related at all. We then need a systematic method of investigating whether there is any relationship between two variables. This leads to the theory of correlations, a simple example of which is given in Sec. 16.

16 | Correlations

In Sec. 15, we discussed the problem of determining the constants in a linear equation relating two variables (in this case \( x \) and \( y \)) by using pairs of observations \((x_i, y_i)\).

of these variables; it was known in advance that such a linear relationship existed.

Sometimes it happens, however, that we do not know in advance whether two variables, say \( x \) and \( y \), are related. Furthermore, if we make pairs of observations \((x_i, y_i)\) as before, the data may be scattered so widely because of experimental errors that it is not clear whether

![Fig. 16.1. To what extent are \( x \) and \( y \) related?](image)

or not there is any relation between \( x \) and \( y \). Representing the observations \((x_i, y_i)\) graphically, we might obtain a picture similar to Fig. 16.1. Are \( x \) and \( y \) related, or are they not? Is there a correlation between \( x \) and \( y \)?

Of course, there is no end to the variety of possible functional relationships between \( x \) and \( y \). There is no general way of investigating all possible relationships,
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but it is fairly easy to check some simple ones. The simplest possible one, of course, is a linear equation. So a reasonable place to start is to ask whether there is a linear relationship between \( x \) and \( y \) or, in other words, a linear correlation.

We can answer this question at least partially by a slight extension of the method of least squares for two unknowns, introduced in Sec. 15. We assume a linear relationship between \( x \) and \( y \) in the form

\[
y = mx + b \quad (16.1)
\]

and proceed to determine the constants \( m \) and \( b \) from the observations \((x_i, y_i)\) in exactly the same manner as in Sec. 15, Eq. (15.11). In particular,

\[
m = \frac{N \Sigma xy - \Sigma x \Sigma y}{N \Sigma x^2 - (\Sigma x)^2} \quad (16.2)
\]

In this expression, and in those which follow, we have dropped the limits of the summation, which are always 1 to \( N \), and also have omitted the summation indices on \( x \) and \( y \). Thus, \( \Sigma xy \) is an abbreviation for \( \sum_{i=1}^{N} x_i y_i \).

The graphical interpretation of the procedure just described is as follows: We are trying to represent the scattered points in Fig. 16.1 by drawing the best straight line through the points. The slope of this line is \( m \), and its intercept on the \( y \) axis is \( b \). Since the deviations we have used in the method of least squares are

\[
d_i = mx_i + b - y_i \quad (16.3)
\]

\( d_i \), represents the vertical distance between the point \((x_i, y_i)\) and the straight line described by constants \( m \) and \( b \). In this case, then, the method of least squares minimizes the sum of the squares of the vertical distances between the point and the straight line. The line determined by this procedure is sometimes called the line of regression of \( y \) on \( x \).

If there is no correlation at all between \( x \) and \( y \), this sum of squares will be minimized by a horizontal line; we shall find therefore in the case of no correlation that \( m = 0 \), a line with zero slope.

Now let us back up slightly. There is no particular reason for writing our assumed linear relationship between \( x \) and \( y \) in the particular form of Eq. (16.1). We might equally well have written instead

\[
x = m'y + b' \quad (16.4)
\]

in which the roles of \( x \) and \( y \) have been reversed. In this case, the deviations we use in the method of least squares are given by

\[
d'_i = m'y_i + b' - x_i \quad (16.5)
\]

The method of least squares now minimizes the sum of the squares of the horizontal distances between the line described by Eq. (16.4) and the points \((x_i, y_i)\) representing the observations. The result is the line of regression of \( x \) on \( y \). The expression for \( m' \) is obtained simply by reversing the roles of \( x \) and \( y \) in Eq. (16.2) and is

\[
m' = \frac{N \Sigma xy - \Sigma x \Sigma y}{N \Sigma y^2 - (\Sigma y)^2} \quad (16.6)
\]
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Note that \( m' \) is not the slope of the line given by Eq. (16.4) but rather its reciprocal. This is easy to see if we solve Eq. (16.4) for \( y \).

\[
y = \frac{x}{m'} - \frac{b'}{m'} \tag{16.7}
\]

We see that the slope of this line is \( 1/m' \), and its intercept with the \( y \) axis is \( -b'/m' \).

Using Eq. (16.4), if there is no correlation between \( x \) and \( y \), the method of least squares will give the value \( m' = 0 \), a vertical line. If, on the other hand, all the points lie exactly on the line, so that the correlation is perfect, then this method must give us the same line as the previous one, Eq. (16.1). That is, in the case of perfect correlation, we must find that \( 1/m' = m \). Thus if there is no correlation between \( x \) and \( y \), \( mm' = 0 \), while if the correlation is perfect, \( mm' = 1 \). Clearly, the value of the product \( mm' \) has something to do with the extent to which the variables \( x \) and \( y \) are correlated.

It is therefore natural to define a correlation coefficient \( r \) as follows:

\[
r = \frac{\sqrt{mm'}}{\sqrt{N \Sigma x^2 - (\Sigma x)^2} \sqrt{N \Sigma y^2 - (\Sigma y)^2}} \tag{16.8}
\]

Thus \( r = 1 \) means perfect correlation, and \( r = 0 \) means no correlation. If there is imperfect correlation, we expect a value of \( r \) somewhere between 0 and 1. In fact, it can be shown that Eq. (16.8) must always have a value between -1 and 1.

Suppose now that we have calculated \( r \) for a set of observations. How do we interpret the result? In other words, how large must \( r \) be in order to indicate a significant correlation between the variables \( x \) and \( y \)? Clearly, because of random fluctuations, we will not in general get exactly \( r = 0 \) even if there is no real connection between \( x \) and \( y \). And if a linear relationship exists, we will not get exactly \( r = 1 \), especially if the experimental errors are large. Given a value of \( r \), then, the question to ask is: What is the probability of obtaining a value of \( r \) as large as this purely by chance from observations on two variables which are not really related? This situation is similar to the one which arose in interpreting the results of a \( \chi^2 \) calculation in Sec. 11.

Tables have been calculated which give the probability of obtaining a given value of \( r \) for various numbers \( N \) of pairs of observations. Table V gives a few values of this probability. A reference to more extensive tables is also given.

Here is an example of the use of this table. Suppose we make 10 observations; then \( N = 10 \). The table says that there is a probability \( P = 0.10 \) of finding a correlation coefficient of 0.549 or larger by chance, and a probability \( P = 0.01 \) of finding \( r \geq 0.765 \), if the variables are not really related. If for our 10 observations we find \( r = 0.9 \), we can be reasonably sure that this indicates a true correlation and not an accident. But if we find only \( r = 0.5 \) we cannot be sure, because there is more than 10% chance that this value will occur by chance.
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A commonly used rule of thumb in interpreting values of $r$ is to regard the correlation as significant if there is less than 1 chance in 20 ($P = 0.05$) that the value will occur by chance. According to this rule of thumb, we find from Table V that for 10 sets of observations, any value of $r$ greater than 0.632 should be regarded as showing a significant correlation. For five sets, $r$ must be greater than 0.878 to be significant.

The theory of correlations can be extended in several directions. First, there may exist a functional relationship between $x$ and $y$ which is not linear and which is not detected by our linear correlation coefficient. For example, if the graph of $x$ versus $y$ is a circle, the correlation coefficient will be zero even if there are no experimental errors. To take such possibilities as this into account, we can assume a quadratic, cubic, or more complicated functional relationship and use the theory of least squares to determine the constants in the equations. Such an analysis gives us nonlinear correlations.

It is also necessary at times to consider correlations among more than two variables, so-called multiple correlations. These extensions of the theory of correlations have rather specialized applications, and we shall not consider them here.

PROBLEMS

2. Find the standard deviation of the mean in Prob. 27, Chap. III. Compare this value with the change in the mean which results from the rejection of unlikely data.

3. A certain quantity was measured $N$ times, and the mean and its standard deviation were computed. If it is desired to increase the precision of the result (decrease $\sigma$) by a factor of 2, how many additional measurements should be made?

4. In Prob. 3, discuss how the mean of the first $N$ measurements should be combined with the mean of the additional measurements, and how the standard deviation of the result should be computed from the standard deviations of the two sets.

5. Show that the standard deviation of a weighted mean is always smaller than any individual standard deviation. Is this a reasonable result?

6. Two different measurements of the speed of light using the same general method (a rotating mirror) yielded the following results:

$$299,796 \pm 4 \text{ km/sec}$$
$$299,774 \pm 2 \text{ km/sec}$$

Are these results consistent? (Assume that the errors given are standard deviations of the means.)

7. In Prob. 13, Chap. I, suppose that the "errors" referred to are standard deviations of the means. Find the standard deviation in $g$. Compare with the result of Prob. 13, Chap. I. Which is more significant?

8. For some obscure reason an individual wants to make an accurate determination of the area of a sheet of typewriting paper. The following measurements are made on the length and width: