Managing the Risk of Deposit Disintermediation
and the Optimal Structure of Input Prices

Abstract
Most bank deposits contain an embedded withdrawal option which permits the depositor to sell the deposit back to the bank at will. Demand deposits generally allow costless withdrawal, while time deposits often require payment of an early withdrawal penalty. Managing the risk that depositors will exercise their withdrawal option is an important aspect of deposit pricing. This paper details the impact of the embedded withdrawal option upon the optimal time deposit rate.
At their most fundamental level banks are in the business of borrowing and lending money. Bank managers are concerned with pricing their assets and their liabilities in order to maximize their profits. While the academic literature on pricing bank assets is vast and well developed, little attention has been given to pricing bank liabilities. In standard banking models, the volume of deposits equate the expected Lerner index of the input price with the inverse of the elasticity of supply to determine the rate paid for deposits. The models have no explicit time dimension so that deposits cannot have any “meaningful” length. Their implicit time to maturity is homogenous across accounts and, without modeling the intertemporal behavior of interest rates, deposits must be assumed to be held until maturity. Commercial banking models clearly ignore two important aspects of input pricing: 1) deposit accounts have different times to maturity, and 2) depositors can withdraw funds before maturity.

Most bank deposits contain an embedded withdrawal option which permits the depositor to sell the deposit back to the bank at will. Demand deposits such as checking, savings, and money market accounts generally allow costless withdrawal, while time deposits often require payment of a pre-specified early withdrawal penalty. Managing the risk that depositors will exercise their withdrawal option is an aspect of deposit pricing. The purpose of this paper is to detail the impact of the embedded withdrawal option upon the optimal time deposit rate. Our objective function is an intertemporal expected cost of deposit funding that begins at \( t_0 \), the inception of the deposit \( D \), and ends at \( T_M \), the time to maturity of the deposit. The model has a benchmark rate of interest whose continuous changes across time are given by trended brownian motion. If the benchmark rate remains below \( r_D \) throughout the time horizon, then the bank’s cost
of funding is simply $r_D \cdot D \cdot (T_m - t_0)$. If the benchmark rate rises above $r_D$ at $T$, then disintermediation occurs. The bank suffers transaction costs $\tau$ and will have to replace the funds at a rate which is higher than the original input price. The determination of the expected cost of deposit funding depends upon the mean time to disintermediation. Using the statistical properties of the benchmark rate of interest we find that time to disintermediation $T$ satisfies Kolmogorov’s diffusion equation. The solution to this differential equation allows us to determine the probability density of the time to disintermediation and its mean.

We take the derivative of the expected cost of deposit funding with respect to $r_D$ and set it equal to zero. Unfortunately, the first order condition for $r_D^*$ is an integral differential equation which refuses to yield an explicit solution for the optimal deposit rate. Not being able to normalize on $r_D^*$ mutes any immediate intuition we might otherwise gain. In addition, analytics provide very little direct insight into the comparative static behavior of deposit pricing. Consequently, our solution for the bank’s optimal input price was simulated over a host of alternative illustrative parameters. Our solution for the input price depends upon the time to maturity, so that $r_D^*$ also yields the optimal structure of time deposit rates.

I. The Time to Disintermediation

Today’s commercial bankers struggle to appropriately price their time deposits. Bankers are convinced that as the economy grows and prospers, interest rates are going to rise. Though anxious to “lock in” deposits at today’s low rates of interest, bankers are aware of the threat of deposit disintermediation in an expansionary economic climate. In the face of this dilemma we propose the following approach to deposit pricing.
Suppose that \( R(t) \) is “the” rate of interest, a kind of average for all rates of interest available on financial assets. Furthermore, we assume that \( R(t) \) will be our benchmark rate of interest, if \( R(t) \) is less than \( r_D \) for all \( t \) over the interval \( [t_0, T_M] \) then no disintermediation will occur and the cost of deposit funding is given by

\[
r_D \cdot D \cdot (T_M - t_0).
\]

However, if the benchmark rate of interest is greater than \( r_D \) at some point \( T_i \) then disintermediation occurs and the cost of funding is given by

\[
r_D \cdot D \cdot (T_i - t_0) + 2\tau + r_i \cdot D \cdot (T_M - T_i).
\]

\( \tau \) is the fixed administrative and decision making costs of providing funds to fleeing depositors and then securing new funds. The variable \( r_i \) is the emergency borrowing rate the bank must pay to replace the disintermediated funds \( (r_i > r_D) \) at time \( T_i \).

Though interest rates in general have no long run trend, we believe it is reasonable to assume that interest rates vary systematically with aggregate economic activity. In particular, interest rates increase over the expansionary phase of the business cycle and fall over the cycle’s contractionary phase.\(^8\) For our model, we will assume that \( R(t) \) follows a brownian motion process with a drift of \( \mu \) where \( \mu > 0 \) since current expectations are that the economy will recover and interest rates will increase.\(^9\)

Examining the expressions (1) and (2) it is clear that determining the expected cost of deposit funding will involve the partial means of two random variables; the cost of emergency borrowing and the time to disintermediation \( T \). The expected cost of emergency borrowing will be no more than the conditional partial expectation of the benchmark rate over the range of \( [r_D, \infty] \). The evaluation of the partial mean of the time
to disintermediation is a little more subtle. However, a convenient statistical relationship exists between our benchmark rate of interest and the time to disintermediation. In particular, since

\[ P\{R(t) < r_D \text{ for } t < T \mid R(0) = r_0\} = \int_{-\infty}^{r_0} p(r_0, r; t) dr \equiv P(r_0, r_D; t). \]

is the probability that the time to disintermediation has not occurred by \( t \), we have

\[ P(r_0, r_D; t) = \text{prob}(T \geq t). \]

So that

\[ P(r_0, r_D; t) = \int_{-\infty}^{r_0} p(r_0, r; t) dr = 1 - G(t; r_0, r_D) \]

where \( G(t; r_0, r_D) \) is the cumulative probability of the time to disintermediation.

Consequently,

\[ \frac{\partial P(r_0, r_D; t)}{\partial t} = -G'(t; r_0, r_D) \]

or

\[ -\frac{\partial P(r_0, r_D; t)}{\partial t} = g(t \mid r_0, r_D) \]

which yields the marginal probability of \( T \).

To solve for \( g(t \mid r_0, r_D) \) we simply take advantage of the fact that since \( p(r_0, r; t) \) is brownian motion it satisfies Kolmogorov’s diffusion equation.\(^{10}\) Since \( P(r_0, r_D; t) \), like \( p(r_0, r; t) \), satisfies the diffusion equation then \( g(t \mid r_0, r_D) \) must satisfy Kolmogorov’s equation, so we have\(^{11}\)

\[ \frac{\partial g(t \mid r_0, r_D)}{\partial t} = \mu \frac{\partial g(t \mid r_0, r_D)}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial^2 g(t \mid r_0, r_D)}{\partial r_0^2}. \quad (3) \]
It is far more convenient to solve for \( g(t \mid r_0, r_D) \) using Laplace transforms with respect to time.

The Laplace transformation changes the partial differential equation above involving the interest rate coordinate \( r \) and the time coordinate \( t \) into an ordinary differential equation in \( r \). Thus we have

\[
\frac{\partial g^*}{\partial t} = \mu \frac{\partial g^*}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial^2 g^*}{\partial r_0 \partial r_0}
\]

where \( g^* = \int_0^\infty e^{-rt} g(t \mid r_0, r_D) dt \) (4)

\[
sg^* = \mu \frac{\partial g^*}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial^2 g^*}{\partial r_0 \partial r_0}
\]

\[
s\phi = \mu \frac{\partial \phi}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial r_0 \partial r_0}
\]

where \( \phi = g^* \).

The expression above is a second order linear differential equation with constant coefficients.

Adopting the trial solution of (4) as \( \phi(r_0) = Ae^{\theta r_0} \) yields \( \phi'(r_0) = \theta Ae^{\theta r_0} \) and \( \phi''(r_0) = \theta^2 Ae^{\theta r_0} \) so that (4) becomes

\[
sAe^{\theta r_0} = \mu \theta Ae^{\theta r_0} + \frac{1}{2} \sigma^2 \theta^2 Ae^{\theta r_0}
\]

\[
Ae^{\theta r_0} (\mu \theta + \frac{1}{2} \sigma^2 \theta^2 - s) = 0
\]

\[
Ae^{\theta r_0} (\frac{1}{2} \sigma^2 \theta^2 + \mu \theta - s) = 0.
\]
Solving \( \left( \frac{1}{2} \sigma^2 \theta^2 + \mu \theta - s \right) = 0 \) for \( \theta \) using the quadratic formula yields

\[
\theta_1(s) \text{ and } \theta_2(s) = \frac{-\mu \mp \sqrt{\mu^2 + 2s\sigma^2}}{\sigma^2}.
\] (5)

The general solution of (4) is

\[
\phi(r_0) = A_1 e^{\theta_1 r_0} + A_2 e^{\theta_2 r_0}
\] (6)

Note that if we take the positive square root in (5), for \( s \) real we have

\[
\theta_1(s) < 0 < \theta_2(s) \quad (s > 0).
\]

In (6), it is clear that \( A_1 \) must be equal to zero because otherwise \( A_1 e^{\theta_1 r_0} \) would be unbounded as \( r_0 \) approaches \( -\infty \) with \( \theta_1(s) \) less than 0.\(^{13} \) Furthermore \( A_2 e^{\theta_2 r_0} \) should be written as \( e^{\theta_2 (r_0 - r_D)} \) so that when \( r_0 = r_D \) , \( \phi(r_0) \) equals one.\(^{14} \) With \( \phi(r_D) = 1 \), the inverse function \( g(s \mid r_0, r_D) \) will be zero.\(^{15} \) This documents a zero likelihood of \( T>0 \) when \( r_0 = r_D \) and establishes the fact that disintermediation will take place immediately.

Finally then, we have

\[
\phi(r_0) = g(s \mid r_0, r_D) = e^{\theta_2 (r_0 - r_D)}(s)
\]

\[
\phi(r_0) = g(s \mid r_0, r_D) = e^{-(s-r_0)(\mu-2\sigma^2)^{1/2})/\sigma^2} \quad \text{for } r_0 < r_D. \] \( (7) \)

We conjecture that

\[
L[g(t; r_0, r_D); s] = e^{-(s-r_0)(\mu-2\sigma^2)^{1/2})/\sigma^2} \] \( (8a) \)

where \( L \) represents the Laplace operator and \( g(t; r_0, r_D) = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} \exp\left[-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}\right]. \)
L[g(t; r₀, r_D); s] could be written as

\[
L[g(t; r₀, r_D); s] = \frac{(r_D - r₀)^{\mu}}{\sigma \sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t^3}} e^{-\frac{1}{2\sigma^2} \left(\frac{(r_D - r₀)^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + 2\sigma^2\right)} e^{-\mu^2 dt} \tag{8b}
\]

or as

\[
L[g(t; r₀, r_D); s] = \frac{(r_D - r₀)^{2\mu} e^{2(r_D - r₀)\mu}}{\sigma \sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t^3}} e^{-\frac{1}{2\sigma^2} \left(\frac{(r_D - r₀)^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + 2\sigma^2\right)} dt \tag{8c}
\]

or as

\[
L[g(t; r₀, r_D); s] = \frac{(r_D - r₀)^{\mu}}{\sigma \sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t^3}} e^{-\frac{1}{2\sigma^2} \left(\frac{(r_D - r₀)^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + 2\sigma^2\right)} \left(\int_0^\infty \frac{1}{\sqrt{t^3}} e^{-\frac{1}{2\sigma^2} \left(\frac{(r_D - r₀)^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + 2\sigma^2\right)} dt \right) \tag{8d}
\]

or, finally, as

\[
L[g(t; r₀, r_D); s] = e^{\frac{(r_D - r₀)^\mu}{\sigma^2}} (r_D - r₀)^{\frac{\mu}{\sigma^2}} \int_0^\infty \frac{1}{\sqrt{t^3}} e^{-\frac{1}{2\sigma^2} \left(\frac{(r_D - r₀)^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + 2\sigma^2\right)} dt \tag{8e}
\]

where \(c = \frac{(r_D - r₀)^2}{2\sigma^2}, \rho = \frac{\mu^2 + 2s\sigma^2}{2\sigma^2}\).

Using the mathematical fact that (see Appendix H)

\[
\int_0^\infty t^{\frac{3}{2}} e^{-c t} e^{-\rho t} dt = \frac{\frac{\pi}{2}}{c} e^{-\frac{\sqrt{c} \rho}{2}}
\]

allows us to rewrite the expression above as

\[
L[g(t; r₀, r_D); s] = e^{\frac{(r_D - r₀)^\mu}{\sigma^2}} (r_D - r₀)^{\frac{\mu}{\sigma^2}} \frac{\sqrt{\pi}}{\sqrt{2\sigma^2}} \frac{\sqrt{\pi}}{\sqrt{2\sigma^2}} e^{-\frac{2(r_D - r₀)^2}{\sqrt{2\sigma^2}} + \frac{2\rho^2}{\sqrt{2\sigma^2}}} \tag{8f}
\]
\[
L[g(t; r_0, r_D); s] = e^{-\frac{\mu(t_0-r_0)}{\sigma^2}} e^{-\frac{(t_0-r_0)(\mu^2+2\sigma^2)}{2\sigma^4}} \quad (8g)
\]

\[
L[g(t; r_0, r_D); s] = e^{\frac{1}{2\sigma^2}(t_0-r_0)(\mu^2+2\sigma^2)} \quad . \quad (8h)
\]

So then indeed our conjecture is correct and the Laplace transform of
\[
g(t; r_0, r_D) = \frac{(r_D-r_0)}{\sigma \sqrt{2\pi t}} \exp\left[\frac{-(r_D-r_0-\mu t)^2}{2\sigma^2 t}\right]
\]
is the right hand side of (8a).

The expression \(g(t; r_0, r_D)\) is then the probability density function of the time to
disintermediation. The probability density function integrates to one and the likelihood of
disintermediation has a mean of \((r_D-r_0)/\mu\). \(^{16}\)

II. The Expected Cost of Deposit Funding

Using equations (1) and (2) we can easily determine the discounted expected cost
of deposit funding, \(E(CDF)\). Recall from equation (1) that if the time to
disintermediation is greater than the time to maturity, \(T_M < T\), then the cost of deposit
funding is simply \(r_D \cdot D \cdot (T_M - t_0)\) so the present value contribution of this possibility to

\[E(CDF)\] would be

\[e^{-\lambda T_M} \cdot r_D \cdot D \cdot T_M \cdot \int_{t_M}^{\infty} g(t) dt \quad \text{for} \quad t_0=0.\]

If \(T < T_M\) then the fixed costs of both disintermediation and deposit refunding are
incurred, \(2\tau\), their discounted inclusion in \(E(CDF)\) would be written as

\[2\tau \cdot \int_{0}^{T_M} e^{-\lambda t} g(t) dt.\]
If the disintermediation occurs at $T_i$ then the cost of deposit funding across the time horizon to $T_M$ would be

$$r_D \cdot D \cdot T_i + r_I \cdot D \cdot (T_M - T_i).$$

Weighted by the likelihood of occurrence and discounted to present value, the expression above would be

$$r_D \cdot D \cdot \int_0^{T_M} e^{-\lambda t} g(t) dt + e^{-\lambda T_M} \cdot D \cdot T_M \cdot \int_{r_D}^{\infty} r f(r) dr - D \cdot \int_0^{T_M} t r e^{-\lambda t} z(t, r) dt dr$$

where $z(t, r)$ is the joint likelihood function of $t$ and $r$.

Resolving the joint likelihood of $z(t, r)$, and collecting the terms above yields the expected cost of deposit funding as

$$E(CDF) = 2r \cdot \int_0^{T_M} e^{-\lambda t} g(t) dt + D \cdot r_D \cdot \int_0^{T_M} t e^{-\lambda t} g(t) dt$$

$$+ e^{-\lambda T_M} \cdot r_D \cdot D \cdot T_M \cdot \int_{r_D}^{\infty} g(t) dt + D \cdot T_M \cdot e^{-\lambda T_M} \cdot \int_{r_D}^{\infty} r \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r-r_0-\mu\Delta)^2}{2\sigma^2\Delta}} dr$$

$$- D \cdot e^{-\lambda T_M} \cdot \int_0^{T_M} t g(t) dt \cdot \int_{r_D}^{\infty} r \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r-r_0-\mu\Delta)^2}{2\sigma^2\Delta}} dr$$

where $\lambda$ is the intertemporal discount factor of the bank.
III. The Optimal Input Price

Minimizing the expected cost of deposit funding with respect to \( r_D \) yields:

\[
2\tau \cdot \frac{\partial}{\partial r_D^*} \left[ e^{-\lambda t_D^*} \int_0^{T_u} t e^{-\lambda t} g(t) dt \right] + \frac{\partial D}{\partial r_D^*} \cdot r_D^* \cdot \int_0^{T_u} t e^{-\lambda t} g(t) dt
\]

\[
+ D \cdot \int_0^{T_u} t e^{-\lambda t} g(t) dt + D \cdot r_D^* \cdot \frac{\partial}{\partial r_D^*} \left[ \int_0^{T_u} t e^{-\lambda t} g(t) dt \right] + e^{-\lambda t_D^*} \cdot T_M \cdot \frac{\partial D}{\partial r_D^*} \cdot \int_0^{T_u} t g(t) dt + e^{-\lambda t_{D*}} \cdot T_M \cdot r_D^* \cdot \frac{\partial D}{\partial r_D^*} \int_0^{T_u} g(t) dt
\]

\[
+ D \cdot T_M \cdot e^{-\lambda t_{D*}} \cdot r_D^* \cdot \frac{\partial}{\partial r_D^*} \left[ \int_0^{T_u} t g(t) dt \right] + e^{-\lambda t_{D*}} \cdot T_M \cdot \frac{\partial D}{\partial r_D^*} \cdot \int_0^{T_u} t g(t) dt + e^{-\lambda t_{D*}} \cdot T_M \cdot D \cdot \frac{\partial D}{\partial r_D^*} \cdot \int_0^{T_u} t g(t) dt
\]

\[
= 0
\]  

(10)

The integral differential equation above refuses to yield an explicit solution for the optimal deposit rate.\(^1\) Examining (10), it is clear that the decision variable arithmetically influences the first order condition in a number of ways. The \( r_D^* \) appears in the limit of five integrals, in ten integrands, in the power to which nearly every exponential function in (10) is raised, and in the determination of the slope of demand for deposits schedule. Given the disparate appearance of the optimal input price in (10), normalizing on \( r_D \) is impossible. Not being able to isolate \( r_D^* \) on the left hand side of such
a complex FOC mutes any immediate intuition. In addition, the complexity of the FOC nearly guarantees that using the implicit function theorem for comparative static analysis will be fruitless. However, it is clear that simulating the bank’s optimal input price would provide us with a good deal of insight.

IV. Some Simulations

In regard to preparing equation (10) for simulation, we first chose a deposit supply equation which is widely used in the literature,

$$D = e^{\alpha r_D}.$$  \hspace{1cm} (11a)

The exponential function above yields a positive relation between $D$ and $r_D$

$$\frac{\partial D}{\partial r_D} = \alpha e^{\alpha r_D} > 0$$ \hspace{1cm} (11b)

suggesting that to increase funds on deposit the bank merely needs to increase $r_D$.\textsuperscript{19} The second derivative is positive and helps insure that the second order condition for $r_D^*$ is satisfied.

The elasticity of supply of bank deposits is given by

$$\frac{\partial D}{\partial r_D} \cdot \frac{r_D}{D} = \alpha r_D,$$ \hspace{1cm} (11c)

and rewriting (11c) we have

$$\frac{\partial D}{\partial r_D} \cdot \frac{1}{D} = \alpha.$$ \hspace{1cm} (11d)

Consequently, dividing the LHS and the RHS of (10) by the level of deposits $D$, we can
replace the term $\frac{\partial D}{\partial r_p} \cdot \frac{1}{D}$ with $\alpha$ wherever it appears in (10). Borrowing from an intermediate level microeconomic textbook, we assume that the RHS of (11c) is less than one.

Scrutinizing (10) it is clear that we still need to specify $\mu, \sigma, T_M, \tau$ and $\lambda$. In order to employ realistic characterizations of the parameters $\mu$ and $\sigma$, we reviewed the behavior of domestic interest rates over the last twenty years. We discovered five intervals of time where short term interest rates increased by at least 200 basis points. The following chart details the rise in the rate of return $\hat{\mu}$ on one year treasury securities and the standard deviation $\hat{\sigma}$ of the rates in each of the five epochs.

(Place Table I here)

We annualize the percentage change in interest rates for each epoch, then weight each $\hat{\mu}$ and $\hat{\sigma}$ according to each epoch’s number of observations relative to the total. Our overall average $\hat{\mu}$ and $\hat{\sigma}$ provides us with the magnitudes for the parameters $\mu$ and $\sigma$.

Obtaining a characterization of the time to maturity $T_M$ for the base case is a little more problematic. We initially computed an estimate of the average time deposit maturity by using data from the 2004 10-K and 10-Q SEC filings for a large sample of commercial banks, which reveal the gross amount of the CD liabilities per year. We derived an estimate of $T_M$ by assigning a value of 1 to all CDs of one year and less, a value of 2 for CDs two years to one year in maturity, etc. We then computed a weighted average time to maturity for the banks in our sample of 1.36 years. However, due to the fact that we do not know whether the CDs mature in 1 day or 365 days within each time interval, our estimate is plagued with what amounts to be an “errors in variables”
problem. Consequently, we have selected four alternative times to maturity that cover the spectrum of CD lengths which are popular at the retail level: 0.5, 1.0, 1.5, and 2.0 years. Serendipitously, this means that instead of having one base case we have four base cases, one for each time to maturity. Our base case solution for \( r^*_p \) provides the maturity structure of the optimal input price and, addresses a major shortcoming of traditional liability pricing models, it recognizes differences in the time to deposit maturity.

Using current (as of November 17, 2004) market data from Bloomberg on the cost of equity, debt, and preferred stock for a large sample of publicly traded banks in the United States, we find a weighted average cost of capital (WACC) of approximately 6%. We then used this WACC as the bank’s intertemporal discount rate, \( \lambda \). The integral differential equation given by (10) was solved numerically using Newton’s search technique for first order conditions. In particular, equation (10) can be written in general as

\[
FOC(r^*_{D,n+1}) = 0
\]

where \( n \) is the number of iterations. Using a first order Taylor expansion on the expression above yields

\[
FOC(r^*_{D,n}) \bigg|_{r^*_D} + FOC'(r^*_{D,n}) \bigg|_{r^*_D} dr^*_{D,n+1} = 0. \tag{12a}
\]

Solving for \( dr^*_{D,n+1} \)

\[
FOC(r^*_{D,n}) \bigg|_{r^*_D} = -FOC'(r^*_{D,n}) \bigg|_{r^*_D} \cdot (r^*_{D,n+1} - r^*_D) \tag{12b}
\]

\[-FOC(r^*_{D,n}) \bigg|_{r^*_D} = FOC'(r^*_{D,n}) \bigg|_{r^*_D} \cdot (r^*_{D,n+1} - r^*_D) \tag{12c}
\]
Newton’s approach to the numeric solution of (10) was chosen for its simplicity and its convergence properties. A tolerance level of $10^{-7}$ was used in the iterations. For the parameters specified above, the expected cost of deposit funding and the optimal $r_D$ is provided below for all four times to maturity associated with the base case.

(insert Figures 1, 2, 3, 4 here)

All four base cases have local minimums and the optimal input prices are all reasonable. Given a 2.37% expected annual appreciation in the benchmark rate and $T_M = 1$ with $r_o = 0.25\%$, $r_D^* = 3.997\%$ seems like a pretty intuitive setting for a decision variable that seeks to balance the probability of suffering the fixed costs of deposit disintermediation with the cost of secure funding from $t_o$ to $T_M$. The fact that $r_D^*$ is increasing with respect to the time to maturity is interesting. An increase in $T_M$ impacts all five RHS terms in (9) and, consequently, its net impact on $E(CDF)$ is impossible to sign analytically. However, insight into the positive relationship between $T_M$ and $r_D$ can be gained by considering the risk of disintermediation, “R of D”. The risk of disintermediation, $\int_{0}^{T_M} g(t) \, dt$, appears in four of the five RHS terms of $E(CDF)$. While $g(t)$ is weighted in 3 of the 4 cases, understanding how $\int_{0}^{T_M} g(t) \, dt$ behaves in the face of parametric changes is useful in understanding how $E(CDF)$ behaves. For example, the partial derivative of the risk of
disintermediation with respect to $T_M$ is easily found to be $\left[ \frac{r_D - r_0}{\sigma \sqrt{2\pi T_M^3}} e^{-\left(\frac{r_D - r_0 - \mu T_M}{\sigma T_M}\right)^2 / 2\sigma^2 T_M^3} \right]$, a positive number. While the partial derivative of $\int_0^{T_M} g(t) \, dt$ with respect to $r_D$ is

$$\left[-\frac{1}{r_D - r_0}\right] \int_0^{T_M} g(t) \left[1 - \frac{(r_D - r_0)^2}{\sigma^2 t} + \frac{\mu(r_D - r_0)}{\sigma^2}\right] dt,$$

a negative number for the parameters used in our simulations. If preserving the existing risk of disintermediation were the bank’s objective function then the implicit function theorem would suggest that $r_D$ will increase with an increase in $T_M$.

$$\frac{dr_D}{dT_M} = \left(-1\right) \frac{\partial R of D}{\partial T_M} \frac{\partial R of D}{\partial r_D} > 0.$$

We believe this kind of intuition can be extended to Diagrams A, B, C, and D. Increasing $T_M$ increases the risk of deposit disintermediation; this increase impacts $E(CDF)$ and, consequently, $r_D^*$ increases to mitigate the impact of increasing the time to maturity.

In order to gain a better understanding of our solution for the optimal structure of input prices, a comparative static analysis was performed. The original parameter settings were alternatively increased by 10% and the results are provided immediately below.

(insert Table II here)
Examining the grid above it is clear that if the expected change in the rate of interest increases, \( r_D^* \) increases throughout the maturity structure. Rather than examining 
\( E(CDF) \) directly, again, we think intuition is well served by considering the risk of deposit disintermediation. The partial derivative of “\( R of D \)” with respect to \( \mu \) is given by
\[
\int_0^{\tau_D} \frac{r_D - r_0}{\sigma \sqrt{2\pi t}} e^{-\frac{(r_D - r_0 - \mu \tau)^2}{2\sigma^2}} \left\{ \frac{(r_D - r_0 - \mu \tau)}{\sigma^2} \right\} dt, \quad \text{a positive number.}
\]
While the \( \frac{\partial R of D}{\partial r_D} \) is negative, the implicit function theorem yields \( \frac{dr_D}{d\mu} > 0 \) for \( d(R of D) = 0 \). An increase in \( \mu \) increases the risk of disintermediation and, consequently, \( r_D \) increases to preserve the existing level of risk of disintermediation. An increase in \( \mu \) increases the risk of disintermediation and adversely impacts \( E(CDF) \) consequently, as documented in the grid, \( r_D^* \) increases across the term to maturity.

The partial derivative of \( \int_0^{\tau_D} g(t) \, dt \) with respect to \( \sigma \) is
\[
\int_0^{\tau_D} g(t) \left\{ \frac{(r_D - r_0 - \mu \tau)^2}{\sigma^2} - \frac{1}{\sigma} \right\} dt, \quad \text{a positive number for all realistic values of } \sigma, \text{ so that an increase in } \sigma \text{ increases the risk of disintermediation. With } \frac{\partial R of D}{\partial r_D} < 0 \text{ then}
\]
\[
\frac{dr_D}{d\sigma} = (-1) \frac{\partial R of D}{\partial \sigma} \frac{\partial \sigma}{\partial R of D} > 0. \text{ An exogenous increase in } \sigma \text{ occasions an increase in the deposit rate if the existing risk of disintermediation is to be maintained. Extending this logic to explain the results in the grid leads us to think an increase in } \sigma \text{ increases the role}
of "R of D" in equation (10) and this occasions the increase in $r_D^*$, as detailed in the grid for all four terms to maturity.

An increase in $\alpha$ essentially amounts to an increase in the elasticity of deposit supply. Consequently, a reduction in the optimal input price across the maturity horizon is not a surprise. Given the nearly identical arithmetic role of $\lambda$ in all five right hand side terms of $E(CDF)$, it is hard to imagine the bank’s intertemporal discount rate having much of an impact upon $r_D^*$. In fact, though an increase in $\lambda$ increases $r_D^*$ in the grid; its impact is nearly imperceivable. The impact of an increase in $\tau$, the fixed administration and decision-making costs of disintermediation and re-intermediation, cannot be understood by considering its impact on "R of D". The risk of disintermediation is independent of the parameter $\tau$; consequently, we must use the implicit function theorem upon the first order condition for the input price and hope that the analytics do not become unmanageable.

In particular,

$$\frac{dr_D^*}{d\tau} = -\frac{\partial FOC(r_D^*)}{\partial \tau} \frac{\partial FOC(r_D^*)}{\partial r_D^*}. $$

Fortuitously the partial derivative of $FOC(r_D^*)$ with respect to $\tau$ is simply

$$\frac{\partial FOC(r_D^*)}{\partial \tau} = 2\frac{\int_0^\tau e^{-\lambda} g(t) dt}{\partial r_D^*}. $$

The RHS of the expression above is the partial derivative of the discounted risk of disintermediation with respect to $r_D^*$ which, according to our earlier analysis must be
negative. Given the convex appearance of $E(CDF)$ in the diagrams above, clearly the second order condition for $r_D^*$ must be satisfied and an optimal input price exists. Since $\partial FOC(r_D^*) / \partial r_D^*$ is the second order condition, it must be positive. The partial derivatives at hand lead us to conclude that $dr_D^* / d\tau$ is positive which is what we have documented for all four optimal deposit prices in Table 2.

V. The Impact of Early Withdrawal Penalties on Deposit Pricing

In pricing their deposit services banks can attach a penalty $\gamma$ for early withdrawal of funds. Though in practice $\gamma$ is often nominal and is frequently waived, if used, it clearly impacts the optimal structure of bank deposit prices. Disintermediation now takes place at benchmark rates that are greater than $r_D + \gamma$ since the benchmark’s yield must now compensate the depositor for the penalty. Consequently, the probability density function of the time to disintermediation is now given by

$$g(t) = \frac{r_D + \gamma - r_0}{\sqrt{2\pi\sigma^2 t^3}} e^{-(t+\gamma-r_0-\mu t)^2 / 2\sigma^2 t}$$

and the conditional density of the benchmark rate at the time of disintermediation is

$$\frac{1}{\sqrt{2\pi\sigma^2 \Delta}} e^{-(r_D + \gamma - \mu \Delta)^2 / 2\sigma^2 \Delta}.$$  

The new expected cost of deposit funding is distinguished from (9) in predictable ways and is given as
Minimizing (13) with respect to \( r_D \) and then using our base case parameters, we solved for the optimal structure of deposit prices with a prepayment penalty.23

(Insert Table III here)

The result above can be understood if we think in terms of managing the risk of disintermediation. The partial derivative of \( \int_0^{T_y} g(t) \, dt \) with respect to \( \gamma \) is

\[
\left. \frac{\partial}{\partial \gamma} \int_0^{T_y} g(t) \, dt \right|_{\gamma=0} = - \int_0^{T_y} \frac{1}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{(r_D + \gamma - r_0 - \mu)^2}{2\sigma^2 t}} \, dt
\]

a negative number. So that if \( d(R of D) \) is to be equal to zero in the face of an increase in \( \gamma \), \( r_D \) must fall.

\[
\frac{dr_D}{d\gamma} = -1 \cdot \frac{\partial R of D}{\partial \gamma} < 0.
\]

An increase in \( \gamma \) reduces the risk of disintermediation and allows the bank to pay less for deposits.

VI. Summary

Today’s commercial bankers are struggling to appropriately price their time deposits.

Bankers are convinced that as the economy grows and prospers, interest rates are going to
rise. Though anxious to lock in deposits at today’s low rates of interest, bankers are aware of the threat of deposit disintermediation in an expansionary economic climate. This paper has outlined a solution to deposit pricing problem faced by bankers. We first proposed a stochastic process to describe the behavior of short term interest rates. The process represented a benchmark rate of interest that signaled alternative returns available to current bank depositors across time. If the benchmark rate moved above \( D_r \), funds would flow out of the bank and into other investment opportunities. Banks would suffer fixed administrative costs and would have to refund its deposits at a rate of interest higher than \( D_r \). If the benchmark rate remained far below \( D_r \) for the deposit horizon, the bank would have probably paid too much for its inputs. We used the prescribed behavior of the benchmark rate to derive the probability density function of the time to disintermediation. From here, we determined the expected cost of deposit funding and minimized it with respect to \( D_r \) to yield the optimal input price. Unable to normalize on the optimal deposit price, we solved for \( D_r^* \) numerically. Furthermore, since the optimal input price is an implicit function of the deposit’s time to maturity we were able to determine the optimal maturity structure of time deposit rates. The comparative static behavior of the structure of input prices was then studied under a host of parametric schemes. In the face of alternative increases in \( \mu, \sigma, \tau, \) and \( \lambda \) the optimal structure of deposit prices shifted upward. An increase in either \( \alpha \) or \( \gamma \) shifted the maturity structure downward. Squeezing any intuition out of the first order condition for the optimal input price was nearly impossible. However, the comparative static behavior of the optimal input prices was easily understood if we thought of \( D_r^* \)’s behavior as trying to maintain the existing
likelihood of deposit disintermediation. Alternative changes in the model’s parameters impact the risk of disintermediation. The subsequent change in the optimal input price should be thought of as trying to preserve the original level of disintermediation risk, that is, of trying to “manage” the risk of deposit disintermediation.
References


Barro, Robert, and Anthony Santomero, 1972, Household Money Holdings and the Demand Deposit Rate, Journal of Money, Banking, and Credit. 4, 397-413.


Footnotes

1 The literature on loan rate determination is extensive and many authors have considered the deposit rate setting behavior of banks including the classic efforts of Klein (1971), Monti (1972), Sealey (1980) and Flannery (1982). Baltensperger (1980), Santomero (1984) and Freixas and Rochet (1997) provide excellent reviews of these two segments of research on commercial banking. Often in the banking literature, pricing of bank outputs and inputs is considered simultaneously in the determination of the optimal intermediation margin. In Ho and Saunders (1981), banks are dealers in loan and deposit markets where the major difficulty in managing the process is that loans and deposits arrive stochastically and not at the same time. The authors show that a bank will charge an intermediation fee for the immediate provision of loan and deposit accounts to its customers. This intermediation margin is shown to be dependent upon management’s degree of risk aversion, the bank’s market structure, the average size of bank transaction, and the variance of interest rates. Subsequent papers have also focused on the optimal intermediation margin as it relates to various types of uncertainty which are common to the banking environment. Wong (1997) analyzes the intermediation margin under interest rate risk and finds it positively related to market power, operating costs, and risk aversion. Wong contrasts his model with others such as Zarruk (1989) where the sole source of uncertainty is funding risk and Zarruk and Madura (1992) where credit is risky. Zarruk (1989) finds that increases in bank capital typically increase the intermediation margin while deposit volatility reduces the margin. Zarruk and Madura (1992) find that
increases in bank capital requirements and deposit insurance premiums reduce borrowing and lending margins. Allen (1988) has analyzed the impact of cross elasticities of bank products upon the intermediation margin; the margin is shown to be dependent on monopoly power, a risk premium and multi-product diversification. Angbazo (1997) has empirically confirmed that banks with riskier loans and higher interest rate risk exposure enjoy larger intermediation margins. Stanhouse and Stock (2004) derive the optimal intermediation margin in a model where loans can be prepaid and deposits can be withdrawn, the simplifying assumptions of their analysis undermines the usefulness of their results. For example, in order to obtain $r_o^*$, Stanhouse and Stock (2004) assume that all deposits have the same implicit time to maturity. Furthermore, they assume that the undeclared time to maturity of deposits coincides with that of bank loans. Finally, they allow the benchmark rate to change just once between recalibrations of the optimal output price and input price.

Please see Appendix A for an illustration of the standard approach to deposit pricing.

Insured time deposits, such as certificates of deposit, provide banks with a significant source of funds. Although banks face increasing competition in all product areas from nonbanks such as finance, investment and insurance companies, the federally insured retail deposit franchise remains unique to the bank charter, thereby offering a potential comparative advantage. In fact as of August 1, 2004, retail CD balances (time deposits less than $100,000) totaled 794.4 billion or 10.1% of assets for commercial banks in the United States. Wholesale CDs (Jumbo CDs, or deposits greater than $100,000) added
another 1042.3 billion or 13.3% of bank assets. Thrift institutions seem to rely even more heavily on CDs. In a recent (6/30/2004) survey of savings institutions by FDIC, total time deposits totaled 343.0 billion dollars or 35.7% of assets.

4 Ahn, Boudoukh, Richardson, and Whitelaw (1999) minimize the firm’s risk by determining the optimal exercise price on options the firm issues. Similarly, this paper could be considered a risk management paper where the bank seeks to manage the risks of early deposit withdrawal. Optimal deposit rates are the optimal striking prices set by the bank as the issuer of the implicit options on deposits.

5 In an already complex model the bank is assumed to face only one supply schedule, therefore only one set of supply elasticities. The bank’s supply schedule is, of course, the consumer’s demand schedule for deposit services. Heffernan (2002), Rossiter (1993), and Hannan (1994) all insist that consumer price elasticities for bank products are different; in particular the interest rate elasticities of depositors for demand deposits and time deposit services are distinct. Having to accommodate the model and use only one input supply schedule, our analysis will focus on time deposits. We chose time deposits partly due to the fact that they are six times larger than demand deposits ($1.818 trillion compared to $310 billion according to current Federal Reserve estimates). But, in addition, the solution yielded by our model would not be directly applicable to demand deposit pricing by commercial banks. In particular, as documented by Heffernan (1992), Barro and Santomero (1972), Rossiter and Lee (1987), and Mitchell (1979), banks “pay” non-pecuniary returns for demand deposits which can take the form of gifts, free
checking, discounted brokerage fees, and a variety of other services. If our model was stylized to compute the optimal input price for demand deposits, the solution would be gross of these implicit payments and would not be immediately useful to a commercial bank.

6 Please see appendix B for a heuristic derivation of the Kolmogorov diffusion equation.

7 Even the most casual reader will recognize that what we call “time to disintermediation” is what statisticians call the “time to first passage”. Please see Bailey (1964), Bartlett (1955), Bharucha-Reid (1960), Darling and Siegert (1953), Doob (1953), Feller (1957), Parzen (1962), and Uhlenbeck and Ornstein (1930) for the background analysis that enabled us to derive the probability density for the time to disintermediation.

8 The notion that interest rates rise during periods of economic recovery and growth is probably justified. Over the last twenty years, we find five periods of time where short term interest rates have risen at least 200 basis points (these epochs are also discussed in the text on page 12 and in footnote #9). The appreciation of the one year rate of return on treasuries during the relevant intervals can be seen in Table I. If the correlation coefficient $\rho$ between the return on treasuries and real gross domestic product is computed for the epochs at hand, the results dramatically endorse the existence of positive relationship between the two variables.

(insert Table IV here)
In order to gain support for characterizing the behavior of short term interest rates as trended brownian motion with $\mu > 0$, we reviewed domestic interest rates over the last twenty years. We discovered five intervals of time where short term rates increased by at least 200 basis points (see Table I) and then tested the appropriateness of a linear trend over the five epochs.

In particular, the mean squared errors (MSE) of linear, quadratic, and exponential trends were computed for one year treasury rates of returns across the length of each interval. For example, during the 2/12/1988-1/27/1989 interval, the three specifications yield the following MSEs:

<table>
<thead>
<tr>
<th></th>
<th>Linear Trend</th>
<th>Quadratic Trend</th>
<th>Exponential Trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE</td>
<td>.027486</td>
<td>.06709</td>
<td>.02926</td>
</tr>
</tbody>
</table>

Clearly the linear entries of the RHS variable fit the data best, which supports our trended brownian motion characterization of interest rates. Similar results hold for the other epochs and for two year treasury rates.

Please see Appendix D for a demonstration of the fact that trended brownian motion satisfies the Kolmogorov diffusion equation.

Please see Appendix E and F for demonstrations that $P(r_0, r_D; t)$ and $g(t | r_0, r_D)$, respectively, satisfy Kolmogorov’s diffusion equation.
Please see Appendix G for a brief tutorial on Laplace transforms.

The benchmark rate of interest has unbounded support in this paper. Consequently, problems of negative interest rates can arise. Throughout the paper, the non-negative constraint is ignored for simplicity. The probability of negative interest rates can be made arbitrarily small by the appropriate choice of the underlying statistical parameters that characterize the density function of $R(t)$.

That is $A_2 = e^{-r_0}$.

Clearly when $r_0 = r_D$, then $\phi(r_D) = 1$. The inverse of $\phi(r_D)$ is

$$g(t; r_0, r_D) = \frac{r_D - r_0}{\sigma \sqrt{2\pi t^3}} \exp\left[-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}\right].$$

If $r_0 = r_D$, the starting value for $r_0$ that made $\phi(r_D)$ equal to 1, we have $g(t; r_0, r_D) = 0$.

Please see appendix I and then appendix J for confirmation that $\int_0^\infty g(t) \, dt$ equals 1 and

$$\int_0^\infty t \, g(t) \, dt = \frac{r_D - r_0}{\mu}.$$

Please see appendix K for the resolution of the joint likelihood $z(t, r)$ into a marginal and a conditional probability density function.
Please see appendix L for the arithmetic details associated with equation (10).

The supply of time deposits schedule is positively sloped so that the bank can secure more deposit funds only by increasing $r_d$ (the bank is assumed to a monopsonist). Our assumption that a bank’s input market is not perfectly competitive is consistent with a large number of published articles. Hannan and Berger (1991) as well as Neumark and Sharpe (1992) report that deposit input prices demonstrate significantly more price rigidity when the stimulus for a price change is upward rather than downward. Adams, Roller and Sickles (2003) find that banks display non-competitive behavior in the deposit market. Sealey and Lindley (1977), Prisman, Slovin and Sushka (1986), Dermine (1986) and Hutchison and Pennacchi (1996) all presume non-competitive bank deposit markets as a point of departure in their respective analyses.

For CDs with terms to maturity longer that the anticipated expansionary phase of the business cycle, our results will be of only limited usefulness.

Using data provided by the Federal Reserve’s Functional Cost Analysis Report (1992) and Koch (1995), $\tau$, administrative costs for small and large CDs, can range between 0.65 and eight cents per dollar.

See Gilkeson, Porter, and Smith (2000) for an alternative analysis of the early withdrawal option.
Please see Appendix M for the arithmetic details associated with the determination of $r^*_D$ when prepayment penalties are involved.
Appendices

Appendix A

Consider the following commercial banking model where the expected profit is given as

\[ E(\pi) = Lr_L - Lc_L - r_D D - c_D D - (L - E - D) \int_k^x f(x) dx \]

and where
- \( L \) = loan size
- \( Lc \) = administrative cost of loan portfolio
- \( D \) = deposit size
- \( Dc \) = administrative cost of deposit liabilities
- \( E \) = equity
- \( r_L \) = rate of return of loan
- \( r_D \) = rate of return on deposits, that is, the input price
- \( x \) is the bank’s cost of borrowing.

Evaluating the expected cost of borrowing yields

\[ E(\pi) = Lr_L - Lc_L - r_D D - c_D D - (L - E - D) \mu_x. \]

Taking the derivative of \( E(\pi) \) with respect to \( r_D \)

\[ \frac{\partial E(\pi)}{\partial r_D} = -D - r_D \frac{\partial D}{\partial r_D} - c_D \frac{\partial D}{\partial r_D} + \mu_x \frac{\partial D}{\partial r_D}. \]

Setting the expression above equal to zero to solve for the optimal price of inputs yields

\[ D + (r_D^* + c_D - \mu_x) \frac{\partial D}{\partial r_D} = 0 \]

\[ (r_D^* + c_D - \mu_x) = -D \frac{1}{\frac{\partial D}{\partial r_D}} \]

\[ r_D^* + c_D - \mu_x = \frac{-1}{\frac{\partial D}{\partial r_D}} \frac{r_D^*}{D} \]

\[ r_D^* + c_D - \mu_x = \frac{1}{\varepsilon} \]
where $\varepsilon$ is equal to the elasticity of supply and the LHS is the familiar Lerner index.

Please note that $r_0^*$ has no explicit time dimension and that the model cannot accommodate the possibility of early withdrawal by depositors.

**Appendix B**

Suppose that the random variable $R$ representing the process in question takes the value $r_0$ at time $t_0$ and $r_1$ at time $t_1$, i.e., $R(t_0) = r_0$ and $R(t_1) = r_1$. Let us write $p(r_0, t_0; r_1, t_1)$ for the conditional frequency function of the variable $r_1$ at time $t_1$ given the value $r_0$ at the previous time $t_0$. Notice that the order of the pairs $(r_0, t_0)$ and $(r_1, t_1)$ represent the direction of the transition. We now consider the three epochs of time $t_0 < t_1 < t_2$, with the corresponding variable values $r_0, r_1, r_2$ where

$$p(r_0, t_0; r_2, t_2) = \int_{-\infty}^{\infty} p(r_0, t_0; r_1, t_1) p(r_1, t_1; r_2, t_2) dr_1$$  \hspace{1cm} (i)

And we are assuming the usual Markov property that, given the present state of the system, the future behavior does not depend on the past. The equation above can be easily proved by considering first a path from $(r_0, t_0)$ to $(r_2, t_2)$ through a particular intermediate point $(r_1, t_1)$. The probability of this specific path for a Markov process is $p(r_0, t_0; r_1, t_1) p(r_1, t_1; r_2, t_2)$. Thus the total probability for a transition from $(r_0, t_0)$ to $(r_2, t_2)$ is obtained by integrating over all possible intermediate points, i.e., integrating with respect to $r_1$ as shown above.

Suppose we consider, for $t_0 < t_1$, the difference
\[ p(r_0, t_0 - \Delta t; r_i, t_i) - p(r_0, t_0; r_i, t_i) \] (ii)

From the expression (i) above, we have

\[ p(r_0, t_0 - \Delta t; r_i, t_i) = \int_{-\infty}^{\infty} p(r_0, t_0 - \Delta t; z, t_0) p(z, t_0; r_i, t_i) \, dz \]

Furthermore, we can write

\[ p(r_0, t_0; r_i, t_i) = p(r_0, t_0; r_i, t_i) \int_{-\infty}^{\infty} p(r_0, t_0 - \Delta t; z, t_0) \, dz \]

because clearly \( \int_{-\infty}^{\infty} p(r_0, t_0 - \Delta t; z, t_0) \, dz = 1. \)

From here we have

\[ p(r_0, t_0 - \Delta t_0; r_i, t_i) - p(r_0, t_0; r_i, t_i) = \int_{-\infty}^{\infty} p(r_0, t_0 - \Delta t_0; z, t_0) [p(z, t_0; r_i, t_i) - p(r_0, t_0; r_i, t_i)] \, dz. \] (iii)

Rewriting the second term in the integrand in terms of a Taylor expansion, yields

\[ \int_{-\infty}^{\infty} p(r_0, t_0 - \Delta t_0; z, t_0) [(z - r_0) \frac{\partial p(r_0, t_0; r_i, t_i)}{\partial r_0} + \frac{1}{2} (z - r_0)^2 \frac{\partial^2 p(r_0, t_0; r_i, t_i)}{\partial^2 r_0 \partial r_0}] \, dz. \]

So that we have

\[ p(r_0, t_0 - \Delta t_0; r_i, t_i) - p(r_0, t_0; r_i, t_i) = \frac{\partial p(r_0, t_0; r_i, t_i)}{\partial r_0} \int_{-\infty}^{\infty} (z - r_0) p(r_0, t_0 - \Delta t_0; z, t_0) \, dz \] (iv)

\[ + \frac{1}{2} \frac{\partial^2 p(r_0, t_0; r_i, t_i)}{\partial^2 r_0 \partial r_0} \int_{-\infty}^{\infty} (z - r_0)^2 p(r_0, t_0 - \Delta t_0; z, t_0) \, dz. \]

Let us now suppose that there exist infinitesimal means and variances for changes in the
basic random variable \( R(t) \) defined by

\[
\mu(r_0, t_0) = \lim_{\Delta t_0 \to 0} \frac{1}{\Delta t_0} \int_{-\infty}^{\infty} (z - r_0) p(r_0, t_0 - \Delta t_0; z, t_0) \, dz
\]

\( \sigma^2(r_0, t_0) = \lim_{\Delta t_0 \to 0} \frac{1}{\Delta t_0} \int_{-\infty}^{\infty} (z - r_0)^2 p(r_0, t_0 - \Delta t_0; z, t_0) \, dz \).

We may have to restrict the ranges of integration on the right hand side of (v) to ensure convergence, but it is unnecessary to pursue this point in our present non-rigorous discussion.

Finally, dividing both sides of (iv) by \( \Delta t \) and letting \( \Delta t \to 0 \) yields:

\[
-\left[ \frac{p(r_0, t_0; r_1, t_1) - p(r_0, t_0 - \Delta t_0; r_1, t_1)}{\Delta t_0} \right] = \frac{\partial p(r_0, t_0; r_1, t_1)}{\partial r_0} \mu(r_0, t_0) + \frac{1}{2} \frac{\partial^2 p(r_0, t_0; r_1, t_1)}{\partial r_0 \partial r_0} \sigma^2(r_0, t_0)
\]

or

\[
-\left[ \frac{\partial p(r_0, t_0; r_1, t_1)}{\partial t_0} \right] = \mu(r_0, t_0) \frac{\partial p(r_0, t_0; r_1, t_1)}{\partial r_0} + \frac{1}{2} \sigma^2(r_0, t_0) \frac{\partial^2 p(r_0, t_0; r_1, t_1)}{\partial r_0 \partial r_0}.
\]

Writing \( p(r_0, r; t) \) as the probability density function of \( r \) at time \( t \) for the time invariant case, we have

\[
-\left[ \frac{\partial p(r_0, r; t)}{\partial t_0} \right] = \mu(r_0, r; t) \frac{\partial p(r_0, r; t)}{\partial r} + \frac{1}{2} \sigma^2(r_0, r; t) \frac{\partial^2 p(r_0, r; t)}{\partial r \partial r}
\]

This then is the backward Kolmogorov diffusion equation for a continuous variable, continuous time stochastic process.
Appendix C

\( f(x) = 3x^2 \) for \( 0 \leq x \leq 1 \); \( F(x) = x^3 \) for \( 0 \leq x \leq 1 \)

\[
P(0 \leq x \leq a) = \int_0^a 3x^2 \, dx = x^3 \bigg|_0^a = a^3 - 0 = a^3
\]

\[
\frac{\partial \text{cdf}}{\partial x} = \frac{\partial F(x)}{\partial x} = 3x^2 = \text{pdf}
\]

Appendix D

Consider \( p(r_0, r; t) = \frac{1}{\sigma \sqrt{2\pi t}} \) e^{-\frac{(r-r_0-\mu t)^2}{2\sigma^2 t}}

\[
\frac{\partial p(\cdot)}{\partial t} = -\frac{1}{2} t^{-3/2} \frac{1}{\sqrt{\sigma^2 2\pi}} e^{-\frac{(r-r_0-\mu t)^2}{2\sigma^2 t}} + \frac{1}{\sigma \sqrt{2\pi t}} \left[ \frac{2(r-r_0-\mu t)\mu [2\sigma^2 t] + 2\sigma^2 (r-r_0-\mu t)^2}{4\sigma^4 t^2} \right]
\]

\[
\frac{\partial p(\cdot)}{\partial r_0} = p(r, r; t) \frac{-2(r-r_0-\mu t)}{2\sigma^2 t} (-1)
\]

\[
\frac{\partial^2 p(\cdot)}{\partial r_0 \partial r_0} = p(r, r; t) \frac{1}{\sigma^4 t^2} \frac{(r-r_0-\mu t)^2}{\sigma^2 t} + p(r, r; t) \frac{-1}{\sigma^2 t}
\]

Substituting into the backward Kolmogorov equation yields (of course since \( t_0 \) has been suppressed we use \( t \) where \( \frac{\partial p(\cdot)}{\partial t} = (-1) \frac{\partial p(\cdot)}{\partial t_0} \))

\[
-\frac{1}{2} t^{-1} p(\cdot) + p(\cdot) \left[ \frac{4(r-r_0-\mu t)\mu \sigma^2 t + 2\sigma^2 (r-r_0-\mu t)^2}{4\sigma^4 t^2} \right]
\]

\[
= p(\cdot) \frac{(r-r_0-\mu t)}{\sigma^2 t} + \frac{1}{2} p(\cdot) \left[ \frac{(r-r_0-\mu t)^2}{\sigma^4 t^2} - \frac{1}{\sigma^2 t} \right] \sigma^2
\]
\[-\frac{1}{2t^2} + \frac{(r-r_0-\mu t)\mu + \frac{1}{2}(r-r_0-\mu t)^2}{\sigma^2 t^2} = \frac{(r-r_0-\mu t)\mu}{\sigma^2 t} + \frac{(r-r_0-\mu t)^2}{2\sigma^2 t^2} = \frac{1}{2t}\]

\[t(r-r_0-\mu t)\mu + \frac{1}{2}(r-r_0-\mu t)^2}{\sigma^2 t^2} = \frac{(r-r_0-\mu t)\mu}{\sigma^2 t} + \frac{(r-r_0-\mu t)^2}{2\sigma^2 t^2}\]

\[\frac{(r-r_0-\mu t)^2}{2\sigma^2 t^2} = \frac{(r-r_0-\mu t)^2}{2\sigma^2 t^2},\]

so \(p(\cdot) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{(r-r_0-\mu)^2}{2\sigma^2 t}}\) satisfies the Kolmogorov diffusion equation.

Appendix E

\[
\frac{\partial P(r_0, r_D; t)}{\partial t} = \mu \frac{\partial P(r_0, r_D; t)}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial^2 P(r_0, r_D; t)}{\partial r_0^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 P(r_0, r_D; t)}{\partial r_0\partial r_0}
\]

because if

\[-\frac{\partial p(r_0, r; t)}{\partial t_0} = \mu \frac{\partial p(r_0, r; t)}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial p(r_0, r; t)}{\partial r_0\partial r_0}\]

then

\[\frac{\partial p(r_0, r; t)}{\partial t} = \mu \frac{\partial p(r_0, r; t)}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial p(r_0, r; t)}{\partial r_0\partial r_0}\]

then
\[
\frac{\partial}{\partial t} \int_{-\infty}^{\eta_0} p(r_0, r; t) dr = \mu \frac{\partial}{\partial r_0} \int_{-\infty}^{\eta_0} p(r_0, r; t) dr + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial r_0 \partial r_0} \int_{-\infty}^{\eta_0} p(r_0, r; t) dr
\]

\[
\frac{\partial P(r_0, r_D; t)}{\partial t} = \mu \frac{\partial P(r_0, r_D; t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 P(r_0, r_D; t)}{\partial r_0 \partial r_0}.
\]

Appendix F

We use \( \frac{\partial g(t \mid r_0, r_D)}{\partial t} \) instead of \( \frac{\partial g(t \mid r_0, r_D)}{\partial t_0} \) because we suppressed \( t_0 \) in the notation.

\[
\frac{\partial g}{\partial t} = \mu \frac{\partial g}{\partial r_0} + \frac{1}{2} \sigma^2 \frac{\partial^2 g}{\partial r_0 \partial r_0}
\]

\[
g = \frac{r_D - r_0}{\sigma \sqrt{2\pi t^3}} e^{-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} = \frac{r^{-3/2} (r_D - r_0)}{\sigma \sqrt{2\pi}} e^{-\frac{(r_D - r_0 - \mu)^2}{2\sigma^2 t}}
\]

\[
\frac{\partial g}{\partial t} = \frac{-\frac{3}{2} t^{-5/2} (r_D - r_0)}{\sigma \sqrt{2\pi}} e^{-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} + \frac{2\mu (r_D - r_0 - \mu t)(2\sigma^2 t) + (r_D - r_0 - \mu t)^2 2\sigma^2}{4\sigma^2 \sigma^2 t^2}
\]

\[
\frac{\partial g}{\partial r_0} = -\frac{1}{\sigma \sqrt{2\pi t^3}} e^{-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} + \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} \frac{r_D - r_0 - \mu t}{\sigma^2 t}
\]

\[
\frac{\partial^2 g}{\partial r_0 \partial r_0} = -\frac{1}{\sigma \sqrt{2\pi t^3}} e^{-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} \frac{[r_D - r_0 - \mu t]}{\sigma^2 t} + \frac{1}{\sigma \sqrt{2\pi t^3}} e^{-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} \frac{[r_D - r_0 - \mu t]}{\sigma^2 t}
\]

We can see that Kolmogorov’s diffusion equation holds because
\[
\frac{\partial g}{\partial t} = -\left(\frac{3}{2}\right) \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} + g\left[ \frac{\mu(r_D - r_0 - \mu t)}{\sigma^2 t} \right] + g\left[ \frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t^2} \right]
\]

\[
\mu \frac{\partial g}{\partial r_0} = -\frac{\mu}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} + \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \frac{(r_D - r_0 - \mu t)}{\sigma^2 t}
\]

\[
\frac{1}{2} \sigma^2 \frac{\partial^2 g}{\partial r_0 \partial \bar{r}_0} = -\frac{1}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{r_D - r_0 - \mu t}{t} \right] + \frac{1}{2} \frac{r_D - r_0}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{(r_D - r_0 - \mu t)^2}{\sigma^2 t^2} \right] - \frac{1}{2} \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{1}{t} \right]
\]

Plugging the right hand side of the three expressions above into the diffusion equation yields:

\[-\left(\frac{3}{2}\right) \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} + g\left[ \frac{\mu(r_D - r_0 - \mu t)}{\sigma^2 t} \right] + g\left[ \frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t^2} \right] =
\]

\[-\frac{\mu}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} + \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \frac{(r_D - r_0 - \mu t)}{\sigma^2 t}
\]

\[-\frac{1}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{r_D - r_0 - \mu t}{t} \right] + \frac{1}{2} \frac{r_D - r_0}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{(r_D - r_0 - \mu t)^2}{\sigma^2 t^2} \right] - \frac{1}{2} \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{1}{t} \right].
\]

So then

\[-\left(\frac{3}{2}\right) \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} + g\left[ \frac{\mu(r_D - r_0 - \mu t)}{\sigma^2 t} \right] =
\]

\[-\frac{\mu}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} + \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \frac{(r_D - r_0 - \mu t)}{\sigma^2 t}
\]

\[-\frac{1}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{r_D - r_0 - \mu t}{t} \right] - \frac{1}{2} \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^3}} e^{(\cdot)} \left[ \frac{1}{t} \right].
\]

then
\[-\frac{3}{2} \frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^2}} e^{(t)} + g\left\{ \frac{\mu (r_D - r_0 - \mu t)}{\sigma^2 t} \right\} = -\frac{\mu}{\sigma \sqrt{2\pi t^3}} e^{(t)} + \mu g\left[ \frac{r_D - r_0 - \mu t}{\sigma^2 t} \right] - \frac{1}{\sigma \sqrt{2\pi t^3}} e^{(t)} \cdot \frac{r_D - r_0 - \mu t}{t} \]

and then

\[-\frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^2}} e^{(t)} + g\left\{ \frac{\mu (r_D - r_0 - \mu t)}{\sigma^2 t} \right\} = -\frac{\mu}{\sigma \sqrt{2\pi t^3}} e^{(t)} + \mu g\left[ \frac{r_D - r_0 - \mu t}{\sigma^2 t} \right] - \frac{1}{\sigma \sqrt{2\pi t^3}} e^{(t)} \cdot \frac{r_D - r_0 - \mu t}{t} \]

After further cancellations we have

\[-\frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^2}} e^{(t)} = -\frac{\mu}{\sigma \sqrt{2\pi t^3}} e^{(t)} - \frac{1}{\sigma \sqrt{2\pi t^3}} e^{(t)} \cdot \frac{r_D - r_0 - \mu t}{t} \]

And finally

\[-\frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^2}} e^{\frac{-(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} = -\frac{(r_D - r_0)}{\sigma \sqrt{2\pi t^2}} e^{\frac{-(r_D - r_0 - \mu t)^2}{2\sigma^2 t}} \]

So we have shown the pdf for the time to disintermediation satisfies the Kolmogorov diffusion equation.

**Appendix G**

The Laplace transform is a method for solving differential equations. For example:
\[ \frac{\partial f(x,t)}{\partial x} + x \frac{\partial f(\cdot)}{\partial t} = 0 \]

\[ \text{L}[\frac{\partial f(x,t)}{\partial x}] + \text{L}[x \frac{\partial f(\cdot)}{\partial t}] = 0 \]

\[ \text{L}[\frac{\partial f(x,t)}{\partial x}] + x \text{L}[\frac{\partial f(\cdot)}{\partial t}] = 0 \]

Then we change the order of the operators on the two terms appearing on the left hand side of the expression above.

\[ \frac{\partial \text{L}(f)}{\partial x} + x[s \text{L}(f) - f(x,0)] = 0 \]

\[ \frac{\partial \text{L}(f)}{\partial x} + x\text{L}(f) = 0. \]

This then we may regard as an ordinary differential equation in \( x \) since derivatives with respect to time do not occur in the equation.

If \( W = \text{L}(f) \) then we have

\[ \frac{\partial W}{\partial x} + x\text{S}W = 0 \]

or we have

\[ W(x,s) = W_0 e^{s x}. \]

From here we have \( W(x,s) = \text{L}(f) \), but to get \( f \) we need to eventually obtain the solution for the Laplace inverse.

Detailing the fourth equation above, by definition \( \text{L}(f') = \int_0^\infty e^{-st} f'(t)dt \).

Using integration by parts \( d(uv) = u \ dv + v \ du \)
\[ uv \bigg|_b^a - \int_b^a u \ dv = \int_b^a v \ du \]

let’s say that

\[ v = e^{-st} \quad du = f'(t) \ dt . \]

So that we have

\[ dv = -se^{-st} \ dt \]
\[ u = f(t). \]

Plugging into our formula

\[ f(t)e^{-st} \bigg|_b^a + s \int_b^a f(t)e^{-st} \ dt \]
\[ f(t)e^{-st} \bigg|_0^\infty + s \int_0^\infty f(t)e^{-st} \ dt \]
\[ -f(t = 0)e^0 + sL(f(t)) = L(f'(t)). \]

Finally, we have

\[ L(f''(t)) = sL(f(t)) - f(t = 0) \]

Consider a Laplace transform

\[ L[f(t); p] = \int_0^\infty e^{-\rho t} f(t) \ dt \]

where \( \rho \) is a positive constant such that \( L[f(t); p] = \int_0^\infty e^{-\rho t} f(t) \ dt \) converges.

Consider an example:
Consider another example:

\[ L[t; p] = \int_{0}^{\infty} te^{-\rho t} dt \]

\[ -\frac{1}{\rho} e^{-\rho t} \bigg|_{0}^{\infty} = \frac{1}{\rho} \]

Consider another example:

\[ L[t; p] = \int_{0}^{\infty} te^{-\rho t} dt \]

\[ d(uv) = u \ dv + v \ du \]

\[ uv \bigg|_{b}^{a} = \int_{b}^{a} v \ du + \int_{b}^{a} u \ dv \]

\[ uv \bigg|_{b}^{a} - \int_{b}^{a} v \ du = \int_{b}^{a} u \ dv \]

\[ u = t \ , \ du = dt \ , \ dv = e^{-\rho t} dt \ , \ \text{and} \ v = -\frac{1}{\rho} e^{-\rho t} \]

\[ -\frac{1}{\rho} e^{-\rho t} \bigg|_{0}^{\infty} + \int_{0}^{\infty} \frac{1}{\rho} e^{-\rho t} dt \]

\[ -\frac{1}{\rho^{2}} e^{-\rho t} \bigg|_{0}^{\infty} = \frac{1}{\rho^{2}} \]

Appendix H

Consider

\[ I(a, b) = \int_{0}^{\infty} e^{-a^{2}u^{2} - b^{2}u^{2}} \ du \]

If \( v = au \), then we can rewrite the expression above as

\[ \int_{0}^{\infty} e^{-v^{2} - \frac{a^{2}}{v^{2}}} \ dv \] or as
\[ I(a,b) = \frac{1}{a} \int_{0}^{\infty} e^{-u^2 - b^2 v^2} \, dv = a^{-1} I(1,ab) \text{ using } v = au. \]

Consider

\[ \frac{\partial I(a,b)}{\partial b} = -2b \int_{0}^{\infty} u^{-\alpha} e^{-u^2 - b^2 v^2} \, du \]

Rewriting the derivative of \( I(a,b) \) with respect to \( b \) in terms of \( v \) where this time \( v = bu^{-1} \), we have \( u^{-2} = \left( \frac{v}{b} \right)^2 \), \( dv = -bu^{-2} \, du \), yields

\[ -2b \int_{\infty}^{0} \left( \frac{v}{b} \right)^2 e^{-u^2 - b^2 v^2} \left(-u^2 b^{-1}\right) dv \]

Keep in mind when \( u = 0, v = \infty \) and when \( u = \infty, v = 0 \)

\[ -2 \int_{0}^{\infty} e^{-u^2 - b^2 v^2} \, dv, \text{ so} \]

\[ \frac{\partial I(a,b)}{\partial b} = -2I(1,ab), \text{ where } v = bu^{-1} \]

\[ I(a,b) = a^{-1} \int_{0}^{\infty} e^{-u^2 - b^2 v^2} \, dv, \text{ where } v = au \]

\[ I(a,b) = a^{-1} I(1,ab), \text{ where } v = au \]

\[ \frac{\partial I(a,b)}{\partial b} = -2I(1,ab), \text{ where } v = bu^{-1} \]

\[ \frac{I(a,b)}{\partial I(\cdot)} = \frac{a^{-1}}{-2} \text{ which implies } -2al(a,b) = \frac{\partial I(a,b)}{\partial b} \]

\[ \frac{\partial I(a,b)}{\partial b} = -2I(1,ab) \]
The solution to the differential equation immediately above is \( I(a, b) = I(a, 0)e^{-2ab} \), where \( I(a, 0) \) is an initial condition.

\[
I(a, 0) = \int_0^\infty e^{-a^2 u^2} \, du = \frac{1}{2} \sqrt{\pi} a^{-1}
\]

\[
I(a, b) = \frac{1}{2} \sqrt{\pi} a^{-1} e^{-2ab}
\]

Finally, we have \( \int_0^\infty e^{-a^2 u^2 - b^2 v^2} \, dv = \frac{\sqrt{\pi}}{2a} e^{-2ab} \)

Briefly,

\[
\int_0^\infty e^{-a^2 u^2 - b^2 v^2} \, dv = a^{-1} \int_0^\infty e^{-y^2 - a^2 b^2 v^2} \, dv, \text{ where } v = au
\]

\[
\frac{\partial}{\partial b} \int_0^\infty e^{-a^2 u^2 - b^2 v^2} \, dv = -2 \int_0^\infty e^{-\nu^2 - a^2 b^2 \nu^2} \, d\nu, \text{ where } \nu = bu^{-1}
\]

\[
\frac{\partial}{\partial b} \int_0^\infty e^{-a^2 u^2 - b^2 v^2} \, dv = -\frac{2}{a} b^{-1}
\]

So we have

\[
\frac{\partial f(\cdot)}{\partial b} = -2af(\cdot)
\]

\[
f(\cdot) = e^{-2ab}, \text{ in particular.}
\]

In general, we have
\[ f(\cdot) = f(b = 0) e^{-2ab} \]
\[ f(\cdot) = \frac{\sqrt{\pi}}{2a} e^{-2ab}. \]

In order to demonstrate the usefulness of this result, let’s consider the integral

\[ \int_0^\infty t^{-1/2} e^{-ct} e^{-\rho t} \, dt. \]

Changing the variable of integration necessitates the following substitutions: \( t = u^2, \ t^{1/2} = u, \ dt = 2u \, du \)

\[ \int_0^\infty t^{-1/2} e^{-c u^2} e^{-\rho u^2} 2u \, du \]

\[ = 2 \int_0^\infty e^{-\rho u^2 - cu^2} \, du \]

\[ = 2 \int_0^\infty e^{-a^2 u^2 - b^2 u^2} \, du \]

where \( a^2 = \rho, \ b^2 = c, \ a = \sqrt{\rho}, \) and \( b = \sqrt{c}. \)

Using the result above

\[ 2 \left[ \frac{\sqrt{\pi}}{2a} e^{-2ab} \right] \]

\[ \left[ \frac{\sqrt{\pi}}{\sqrt{\rho}} e^{-2\sqrt{\rho c}} \right] \]

\[ L[t^{-1/2} e^{-c t}; \rho] = \frac{\sqrt{\pi}}{\sqrt{\rho}} e^{-2\sqrt{\rho c}} \]

Consider another very important result. Take the derivative of the left hand side and the right hand side of the previous expression with respect to \( c. \)

\[ L[t^{-1/2} (-c) e^{-c t}; \rho] = \frac{\sqrt{\pi}}{\sqrt{\rho}} e^{-2\sqrt{\rho c}} (-2) \left( \frac{1}{2} \right) (c \rho)^{-1/2} \rho \]
Be sure to multiply the LHS and the RHS by (-1)

\[ L[t^{-3/2} e^{-c/t}; \rho] = \frac{\sqrt{\pi}}{\sqrt{c}} e^{2\sqrt{\rho c}} \frac{1}{\sqrt{c} \sqrt[4]{\rho}} \]

= \frac{\sqrt{\pi}}{\sqrt{c}} e^{-2\sqrt{\rho c}}

If it was conjectured that

\[ \int_{0}^{\infty} e^{-a^2 u^2} du = \frac{1}{2} \sqrt{\pi} a^{-1} \]

would I believe this?

I know that

\[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 \]

\[ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi} \]

\[ \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{\pi} \frac{\sqrt{2}}{2} \]

\[ \int_{0}^{\infty} e^{-\frac{1}{2\pi} z^2} dz = \frac{1}{2} \sqrt{\pi} \sqrt{2}, \text{ therefore} \]

\[ \int_{0}^{\infty} e^{-a^2 z^2} dz = \frac{1}{2} \sqrt{\pi} a^{-1}, \text{ where } a = \frac{1}{\sqrt{2}} \]

so I would believe the conjecture.
Appendix I

Does \( \int_0^\infty g(t)dt \) equal 1?

\[
\int_0^\infty g(t) \ dt = \int_0^\infty \left( \frac{r_D - r_0}{\sigma \sqrt{2\pi}} \right) \exp\left[ -\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t} \right] \ dt
\]

\[
\int_0^\infty g(t) \ dt = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \int_0^\infty t^{-3/2} e^{-\frac{(r_D - r_0)^2 + \mu^2 t^2 - 2\mu t (r_D - r_0)}{2\sigma^2 t}} \ dt
\]

\[
\int_0^\infty g(t) \ dt = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \int_0^\infty t^{-3/2} e^{-\frac{1}{2\sigma^2 t} \left( (r_D - r_0)^2 + \mu^2 t^2 \right)} \ dt
\]

\[
\int_0^\infty g(t) \ dt = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \int_0^\infty t^{-3/2} e^{-\frac{1}{2\sigma^2 t} \left( (r_D - r_0)^2 + \mu^2 t^2 \right)} \ dt
\]

\[
\int_0^\infty g(t) \ dt = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \int_0^\infty t^{-3/2} e^{-\frac{1}{2\sigma^2 t} \left( (r_D - r_0)^2 + \mu^2 t^2 \right)} \ dt
\]

where \( c = \frac{(r_D - r_0)^2}{2\sigma^2}, \rho = \frac{\mu^2}{2\sigma^2} \). According to Appendix H

\[
\int_0^\infty t^{-3/2} e^{-\frac{1}{2\sigma^2 t} \left( (r_D - r_0)^2 + \mu^2 t^2 \right)} \ dt = \frac{\sqrt{\pi}}{c} e^{-\sqrt{\rho c}}\rho so we have
\]

\[
\int_0^\infty g(t) \ dt = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \int_0^\infty \sqrt{\frac{\pi}{(r_D - r_0)^2}} e^{-\frac{2\sigma^2 \pi}{2\sigma^2}} \ dt
\]

\[
\int_0^\infty g(t) \ dt = \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \int_0^\infty \frac{2\sigma^2 \pi}{r_D - r_0} e^{-\frac{1}{2\sigma^2} (r_D - r_0)^2} \ dt
\]

\[
\int_0^\infty g(t) \ dt = 1
\]
Appendix J

Let’s check out the mean:

\[ \int_{0}^{\infty} t g(t) dt = \int_{0}^{\infty} \frac{(r_D - r_0)}{\sigma \sqrt{2\pi}} \exp\left[-\frac{(r_D - r_0 - \mu t)^2}{2\sigma^2 t}\right] dt \]

\[ \int_{0}^{\infty} t g(t) dt = \frac{(r_D - r_0)^2}{\sigma \sqrt{2\pi}} \int_{0}^{\infty} t^{-1/2} e^{-\frac{1}{2\sigma^2 t} - \frac{1}{2} (r_D - r_0)^2} dt \]

\[ \int_{0}^{\infty} t g(t) dt = \frac{(r_D - r_0)^2}{\sigma \sqrt{2\pi}} \int_{0}^{\infty} t^{-1/2} e^{-\frac{1}{2\sigma^2 t} + \frac{1}{2} (r_D - r_0)^2} dt \]

where \( c = \frac{(r_D - r_0)^2}{2\sigma^2} \), \( \rho = \frac{\mu^2}{2\sigma^2} \).

available in Appendix H is the mathematical fact that

\[ \int_{0}^{\infty} t^{-1/2} e^{-ct} e^{\sigma^2 t} dt = \frac{\pi}{\sqrt{\rho}} e^{-2(c\rho)^{1/2}} \] so that

\[ \int_{0}^{\infty} t g(t) dt = \frac{(r_D - r_0)^2}{\sigma \sqrt{2\pi}} \frac{(r_D - r_0)^2}{\sigma \sqrt{2\pi}} \frac{2\sqrt{2\sigma^2}}{\sqrt{\mu^2}} \frac{\sqrt{2\sigma^2}}{\sqrt{\mu^2}} \frac{\sqrt{2\sigma^2}}{\sqrt{\mu^2}} \]

\[ \int_{0}^{\infty} t g(t) dt = \frac{(r_D - r_0)}{\mu} \]
Appendix K

The partial expectation of the joint probability density function

\[ \int_{t_0}^{\infty} \int_{r_0}^{\infty} e^{-\lambda r} t \, z(t, r) \, dr \, dt \]

can be rewritten as

\[ \int_{t_0}^{\infty} \int_{r_0}^{\infty} e^{-\lambda r} t \, g(t) \, f(r) \, dr \, dt \]

as soon as it is realized that at the time of disintermediation \( T_i \), when the benchmark rate equals or exceeds \( r_p \), another clock starts. This new clock is the reaction time \( \Delta \) of depositors to this new opportunity of disintermediation, let’s say 4 hours or

\[
\frac{4}{(24 \cdot 7 \cdot 52)} = \frac{1}{2184}
\]

depositors to this new opportunity of disintermediation, let’s say 4 hours or

\[
\frac{1}{2184}
\]

of a year. Conditional on passage of the benchmark rate through \( r_p \) and \( \Delta \), the probability density function of the benchmark rate is given as

\[
\frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r-r_0-\mu \Delta)^2}{2\sigma^2\Delta}}
\]

So that we now have

\[
\int_{t_0}^{\infty} \int_{r_0}^{\infty} e^{-\lambda r} t \, \frac{(r_D - r_0)}{\sigma\sqrt{2\pi t}} e^{-\frac{(r_D - r_0 - \mu \Delta)^2}{2\sigma^2 t}} \cdot \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r-r_0-\mu \Delta)^2}{2\sigma^2\Delta}} \, dt \, dr
\]

\[
\int_{t_0}^{\infty} \int_{r_0}^{\infty} e^{-\lambda t} g(t) \, dt \left[ \int_{r_0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r-r_0-\mu \Delta)^2}{2\sigma^2\Delta}} \, dr \right] \]

\[
\int_{0}^{\infty} e^{-\lambda t} g(t) \, dt \left[ \int_{r_0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r-r_0-\mu \Delta)^2}{2\sigma^2\Delta}} \, dr \right] \]

which is what we have in the paper.
Appendix L

Now let’s explicate some of the partial moments for further differentiation, recall

\[ g(t) = \frac{r_D - r_0}{\sigma \sqrt{2\pi t}^3} e^{-\frac{(r_D - r_0)^2}{2\sigma^2 t}}. \]

From here I can show that

\[ \int_0^\infty g(t) \, dt = 1 \]

and that

\[ \int_0^\infty t \, g(t) \, dt = \frac{r_D - r_0}{\mu} \]

\[ \int_0^\infty e^{-\lambda t} \, g(t) \, dt = \int_0^\infty \frac{r_D - r_0}{\sqrt{2\pi t}^3} e^{\frac{-(r_D - r_0)^2}{2\sigma^2 t}} \, dt \]

\[ \int_0^\infty t \, e^{-\lambda t} \, g(t) \, dt = \int_0^\infty \frac{r_D - r_0}{\sqrt{2\pi t}^3} e^{\frac{-(r_D - r_0)^2}{2\sigma^2 t}} \, dt \]

\[ \int_0^\infty t \, g(t) \, dt = \int_0^\infty \frac{r_D - r_0}{\sigma \sqrt{2\pi t}^3} e^{\frac{-(r_D - r_0)^2}{2\sigma^2 t}} \, dt \]

Now the remaining derivatives:

\[ \frac{\partial}{\partial r_D} \int_0^{\tau_D} e^{-\lambda t} \, g(t) \, dt \]

\[ = \int_0^{\tau_D} \frac{1}{\sigma \sqrt{2\pi t}^3} e^{\frac{-(r_D - r_0)^2}{2\sigma^2 t}} \, dt \]

\[ + \int_0^{\tau_D} \frac{r_D - r_0}{\sigma \sqrt{2\pi t}^3} e^{\frac{-(r_D - r_0)^2}{2\sigma^2 t}} \, \left[ \frac{-\lambda (r_D - r_0 - \mu t)}{\sigma^2 t} \right] \, dt \]}
\[
\frac{\partial}{\partial r_D} \left[ \int_{\tau_D}^{\infty} g(t) dt \right] = \int_{\tau_D}^{\infty} \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-\frac{(r_0 - r_D - \mu)^2}{2\sigma^2 t}} dt + \int_{\tau_D}^{\infty} \frac{(r_D - r_0)}{\sqrt{2\pi t\sigma^2}} e^{-\frac{(r_0 - r_D - \mu)^2}{2\sigma^2 t}} \cdot \frac{-(r_D - r_0 - \mu)}{\sigma^2 t} dt
\]

\[
\frac{\partial}{\partial r_D} \left[ \int_{0}^{\tau_D} t e^{-\lambda t} g(t) dt \right] = \int_{0}^{\tau_D} \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(r_0 - r_D - \mu)^2 - 2\lambda\sigma^2 t}{2\sigma^2 t}} dt + \int_{0}^{\tau_D} \frac{(r_D - r_0)}{\sqrt{2\pi t\sigma^2}} e^{-\frac{(r_0 - r_D - \mu)^2 - 2\lambda\sigma^2 t}{2\sigma^2 t}} \cdot \frac{-(r_D - r_0 - \mu)}{\sigma^2 t} dt
\]

\[
\frac{\partial}{\partial r_D} \left[ \int_{r_0}^{\infty} \frac{1}{\sqrt{2\pi \sigma^2 \Delta}} e^{-\frac{(r-r_D - \mu\Delta)^2}{2\sigma^2 \Delta}} dr \right] = \frac{-r_D}{\sqrt{2\pi \sigma^2 \Delta}} e^{-\frac{\Delta^2}{2\sigma^2 \Delta}}
\]

\[
+ \int_{r_0}^{\tau_D} \frac{1}{\sqrt{2\pi \sigma^2 \Delta}} e^{-\frac{(r-r_D - \mu\Delta)^2}{2\sigma^2 \Delta}} \cdot \frac{(r-r_D - \mu\Delta)}{\sigma^2 \Delta} dr
\]

\[
\frac{\partial}{\partial r_D} \left[ \int_{0}^{\tau_D} t g(t) dt \right] = \int_{0}^{\tau_D} \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(r_0 - r_D - \mu)^2}{2\sigma^2 t}} dt + \int_{0}^{\tau_D} \frac{(r_D - r_0)}{\sqrt{2\pi t\sigma^2}} e^{-\frac{(r_0 - r_D - \mu)^2}{2\sigma^2 t}} \cdot \frac{-(r_D - r_0 - \mu)}{\sigma^2 t} dt
\]
Operationalizing the first order condition for simulation, using $D = e^{\alpha r}$ and $\frac{\partial D}{\partial r} \cdot \frac{1}{D} = \alpha$ yields:

\[
\alpha \cdot FC \cdot \int_0^{T_u} e^{-\lambda t} g(t) dt + FC \cdot \frac{\partial r^*_D}{\partial r^*_D} \cdot \int_0^{T_u} t e^{-\lambda t} g(t) dt
\]

\[
+ \int_0^{T_u} t e^{-\lambda t} g(t) dt + r^*_D \cdot \frac{\partial r^*_D}{\partial r^*_D}
\]

\[
+ e^{-\lambda T_u} T_M \int_0^{T_u} g(t) dt + e^{-\lambda T_u} T_M r^*_D \cdot \alpha \int_0^{T_u} g(t) dt + T_M \cdot e^{-\lambda T_u} \cdot r^*_D \cdot \frac{\partial r^*_D}{\partial r^*_D}
\]

\[
+ e^{-\lambda T_u} T_M \alpha \int_0^{\infty} r \left[ \frac{1}{\sqrt{2\pi\sigma^2 \Delta}} \right] e^{-\frac{(r - \mu - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[
+ e^{-\lambda T_u} T_M \alpha \int_0^{\infty} r \left[ \frac{1}{\sqrt{2\pi\sigma^2 \Delta}} \right] e^{-\frac{(r - \mu - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[
- \alpha e^{-\lambda T_u} \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \Delta}} e^{-\frac{(r - \mu - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[
- e^{-\lambda T_u} \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \Delta}} e^{-\frac{(r - \mu - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[
= 0
\]
Appendix M

Minimizing the expected cost of deposit funding with respect to \( r_D \) (under the assumption of a prepayment penalty) yields:

\[
2\tau \cdot \frac{\partial}{\partial r_D^*} \left[ e^{-\lambda t} g(t) dt \right] + D \left[ e^{-\lambda t} g(t) dt + D \cdot (r_D^* - \gamma) \right] + \frac{\partial D}{\partial r_D^*} \left[ e^{-\lambda t} g(t) dt \right] + D \cdot e^{-\lambda T_M} \cdot D \cdot T_M \cdot e^{-\lambda T_M} \cdot r_D^* \cdot \frac{\partial D}{\partial r_D^*} \int_0^{T_M} g(t) dt + e^{-\lambda T_M} \cdot T_M \cdot r_D^* \cdot \frac{\partial D}{\partial r_D^*} \int_{T_M}^{\infty} g(t) dt
\]

Now let’s explicate some of the partial moments for further differentiation, keeping in mind that the evaluations will be impacted by \( \gamma \).

\[
g(t) = \frac{r_D + \gamma - r_0}{\sigma\sqrt{2\pi t}} e^{-\frac{(r_D + \gamma - r_0 - \mu t)^2}{2\sigma^2 t}}
\]
From here I can show that

\[ \int_{0}^{\infty} g(t) \, dt = 1 \]

and that

\[ \int_{0}^{\infty} t \, g(t) \, dt = \frac{r_{D} + \gamma - r_{0}}{\mu} \]

\[ \int_{0}^{\infty} e^{-\lambda t} \, g(t) \, dt = \int_{0}^{\infty} \frac{r_{D} + \gamma - r_{0}}{\sqrt{2 \pi t^{3} \sigma^{2}}} e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t} \, dt \]

\[ \int_{0}^{\infty} t \, e^{-\lambda t} \, g(t) \, dt = \int_{0}^{\infty} \frac{r_{D} + \gamma - r_{0}}{\sqrt{2 \pi t^{3} \sigma^{2}}} e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t} \, dt \]

\[ \int_{0}^{\infty} t \, g(t) \, dt = \int_{0}^{\infty} \frac{r_{D} + \gamma - r_{0}}{\sigma \sqrt{2 \pi t}} e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t} \, dt \]

Now the remaining derivatives:

\[ \frac{\partial}{\partial r_{D}} \left[ \int_{0}^{T_{u}} e^{-\lambda t} \, g(t) \, dt \right] = \int_{0}^{T_{u}} \frac{1}{\sigma \sqrt{2 \pi t^{3}}} \frac{e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t}}{\sigma^{2} t} \, dt \]

\[ + \int_{0}^{T_{u}} \frac{r_{D} + \gamma - r_{0}}{\sigma \sqrt{2 \pi t^{3}}} e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t} \left[ \frac{-(r_{D} + \gamma - r_{0} - \mu t)}{\sigma^{2} t} \right] \, dt \]

\[ \frac{\partial}{\partial r_{D}} \left[ \int_{0}^{T_{u}} g(t) \, dt \right] = \int_{0}^{T_{u}} \frac{1}{\sqrt{2 \pi t^{3} \sigma^{2}}} \frac{e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t}}{\sqrt{2 \pi t^{3} \sigma^{2}}} \, dt + \int_{0}^{T_{u}} \frac{(r_{D} + \gamma - r_{0})}{\sqrt{2 \pi t^{3} \sigma^{2}}} \frac{e^{-(r_{0} + \gamma - r_{0} - \mu t)^{2}/2 \sigma^{2} t}}{\sqrt{2 \pi t^{3} \sigma^{2}}} \left[ \frac{-(r_{D} + \gamma - r_{0} - \mu t)}{\sigma^{2} t} \right] \, dt \]
\[ \frac{\partial}{\partial r_D} \left[ \int_0^{T_D} t e^{-\gamma g(t)} dt \right] = \int_0^{T_D} \frac{1}{\sigma \sqrt{2\pi t}} e^{-\left(\frac{\left(r_D + \gamma - \gamma_0 - \mu t\right)^2}{2\sigma^2 t}\right)} dt \]

\[ + \int_0^{T_D} \frac{\left(r_D + \gamma - \gamma_0\right)}{\sqrt{2\pi t\sigma^2}} e^{-\left(\frac{\left(r_D + \gamma - \gamma_0 - \mu t\right)^2}{2\sigma^2 t}\right)} \left[-\frac{(r_D + \gamma - r_0 - \mu t)}{\sigma^2 t}\right] dt \]

\[ \frac{\partial}{\partial r_D} \left[ \int_{r_0 + \gamma}^{r_D} r \left[\frac{1}{\sqrt{2\pi \sigma^2 \Delta}} e^{\frac{-\left(\left(r - (r_D + \gamma) - \mu \Delta\right)^2}{2\sigma^2 \Delta}} dr \right] \right] = \left(-1\right) \frac{r_D + \gamma}{\sqrt{2\pi \sigma^2 \Delta}} e^{-\frac{(r_D + \gamma)^2}{2\sigma^2 \Delta}} \]

\[ + \int_{r_0 + \gamma}^{r_D} r \left[\frac{1}{\sqrt{2\pi \sigma^2 \Delta}} e^{\frac{-\left(\left(r - (r_D + \gamma) - \mu \Delta\right)^2}{2\sigma^2 \Delta}} \cdot \frac{(r - (r_D + \gamma) - \mu \Delta)}{\sigma^2 \Delta} \right] dr \]

\[ \frac{\partial}{\partial r_D} \left[ \int_0^{T_D} t e^{-\gamma g(t)} dt \right] = \int_0^{T_D} \frac{1}{\sigma \sqrt{2\pi t}} e^{-\left(\frac{\left(r_D + \gamma - \gamma_0 - \mu t\right)^2}{2\sigma^2 t}\right)} dt \]

\[ + \int_0^{T_D} \frac{\left(r_D + \gamma - \gamma_0\right)}{\sqrt{2\pi t\sigma^2}} e^{-\left(\frac{\left(r_D + \gamma - \gamma_0 - \mu t\right)^2}{2\sigma^2 t}\right)} \left[-\frac{(r_D + \gamma - r_0 - \mu t)}{\sigma^2 t}\right] dt \]
Operationalizing the FOC, using \( D = e^{\alpha t_0} \) and \( \frac{\partial D}{\partial r_D} \cdot \frac{1}{D} = \alpha \), yields:

\[
\alpha \cdot 2\tau \cdot \int_0^{T_u} e^{-\lambda t} g(t) dt + FC \cdot \frac{\partial }{\partial r_D^*} \int_0^{T_u} t e^{-\lambda t} g(t) dt + \alpha (r_D^* - \gamma) \int_0^{T_u} t e^{-\lambda t} g(t) dt
\]

\[
+ \int_0^{T_u} t e^{-\lambda t} g(t) dt + (r_D^* - \gamma) \frac{\partial }{\partial r_D^*} \int_0^{T_u} t e^{-\lambda t} g(t) dt
\]

\[
+ e^{-\lambda T_u} T_M \int_0^{\infty} g(t) dt + e^{-\lambda T_u} T_M r^*_D \int_0^{\infty} g(t) dt + T_M \cdot e^{-\lambda T_u} \cdot r^*_D \frac{\partial }{\partial r_D^*} \int_0^{\infty} g(t) dt
\]

\[
+ e^{-\lambda T_u} T_M \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r - (r_D^* + \gamma) - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[
+ e^{-\lambda T_u} T_M \frac{\partial }{\partial r_D^*} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r - (r_D^* + \gamma) - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[-\alpha e^{-\lambda T_u} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r - (r_D^* + \gamma) - \mu\Delta)^2}{2\sigma^2\Delta}} dr
\]

\[
- e^{-\lambda T_u} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r - (r_D^* + \gamma) - \mu\Delta)^2}{2\sigma^2\Delta}} dr \cdot \frac{\partial }{\partial r_D^*} \int_0^{T_u} g(t) dt
\]

\[-e^{-\lambda T_u} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r - (r_D^* + \gamma) - \mu\Delta)^2}{2\sigma^2\Delta}} dr \cdot \frac{\partial }{\partial r_D^*} \int_0^{T_u} g(t) dt
\]

\[-e^{-\lambda T_u} \int_0^{T_u} g(t) dt \cdot \frac{\partial }{\partial r_D^*} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta}} e^{-\frac{(r - (r_D^* + \gamma) - \mu\Delta)^2}{2\sigma^2\Delta}} dr = 0
\]
Appendix N

Leibniz’s rule:

Consider the derivative of an integral where the limits are a function of the variable of differentiation.

\[
y = \int_{g(x)}^{h(x)} f(x) \, dx
\]

\[
\frac{\partial y}{\partial x} = \int_{g(x)}^{h(x)} f'(x) \, dx + \frac{\partial h(x)}{\partial x} \cdot f(h(x)) - \frac{\partial g(x)}{\partial x} \cdot f(g(x))
\]

Appendix O

For example, consider the probability density function for trended brownian motion

\[
p(r_0, r; t_0, t) = \frac{1}{\sigma \sqrt{2\pi(t-t_0)}} e^{-\frac{(r-r_0 - \mu(t-t_0))^2}{2\sigma^2(t-t_0)}}
\]

by observation \( t_0 \) enters \( p(r_0, r; t_0, t) \) as the mirror reflection of \( t \) and, consequently,

\[
\frac{\partial p(\cdot)}{\partial t_0} = -\frac{\partial p(\cdot)}{\partial t}.
\]
Appendix P

When most of us “hear” trended brownian motion, we think

\[ dr_u = \mu du + \sigma dW_u . \]

However, integrating across the time horizon we have

\[
\int_0^t dr_u = \int_0^t \mu du + \sigma \int_0^t dW_u
\]

\[ r_t - r_0 = \mu(t-0) + \sigma \int_0^t dW_u \]

\[ r_t = r_0 + \mu t + \sigma \int_0^t dW_u \]

\[ E(r_t) = r_0 + \mu t \]

\[ V(r_t) = \sigma^2 V\left[ \int_0^t dW_u \right] \]

\[ V(r_t) = \sigma^2 E\left[ \int_0^t dW_u \right]^2 . \]

Ito’s isometry allows us to write

\[ E[I\left[ \int_0^t dW_u \right]\left[ \int_0^t dW_u \right]] \text{ as } \int_0^t du \text{, so we have} \]

\[ V(r_t) = \sigma^2 \int_0^t du \]

\[ V(r_t) = \sigma^2 t \]

\[ r_t \sim N(r_0 + \mu t, \sigma^2 t). \]

with these characteristics the pdf for the left hand side variable \( r_t \) must be as follows

\[ p(r_0, r; t) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{(r-r_0-\mu t)^2}{2\sigma^2 t}} . \]
Table I: Rises in one-year interest rates

This table details recent increases in the one-year treasury interest rates over five different time periods. Also included are the annualized increases, $\hat{\mu}$, and their standard deviation, $\hat{\sigma}$, for each epoch. In each case the one-year rates increased more than 200 basis points.

<table>
<thead>
<tr>
<th>Date</th>
<th># of months</th>
<th>Δbps</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8/22/86-10/16/87</td>
<td>14</td>
<td>248</td>
<td>.0213</td>
<td>.0061</td>
</tr>
<tr>
<td>2/12/88-1/27/89</td>
<td>12</td>
<td>238</td>
<td>.0238</td>
<td>.0078</td>
</tr>
<tr>
<td>10/8/93-5/13/94</td>
<td>7</td>
<td>214</td>
<td>.0367</td>
<td>.0102</td>
</tr>
<tr>
<td>6/10/94-12/30/94</td>
<td>6</td>
<td>205</td>
<td>.0410</td>
<td>.0127</td>
</tr>
<tr>
<td>10/23/98-5/19/00</td>
<td>17</td>
<td>239</td>
<td>.0151</td>
<td>.0042</td>
</tr>
<tr>
<td>Average $\hat{\mu}$</td>
<td></td>
<td></td>
<td>.0237</td>
<td></td>
</tr>
<tr>
<td>Average $\hat{\sigma}$</td>
<td></td>
<td></td>
<td></td>
<td>.0070</td>
</tr>
</tbody>
</table>
TABLE II: Optimal Input Prices

At the top of the table are the base case optimal time deposit rates for a given time to maturity, $T_M$, using the following parameters:

$\mu = 0.0237, \sigma = 0.007, \alpha = 1.0, \tau = 0.10, \lambda = 0.06.$

Below these optimal rates are the changes in the $r_D^*$ given alternative ten percent increases in the parameters at hand.

<table>
<thead>
<tr>
<th>Base case</th>
<th>$r_D^* = 0.02687$</th>
<th>$r_D^* = 0.03997$</th>
<th>$r_D^* = 0.05087$</th>
<th>$r_D^* = 0.05922$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T_M = 0.5$</td>
<td>$T_M = 1.0$</td>
<td>$T_M = 1.5$</td>
<td>$T_M = 2.0$</td>
</tr>
<tr>
<td>$\Delta r_D^*$</td>
<td>$\Delta r_D^*$</td>
<td>$\Delta r_D^*$</td>
<td>$\Delta r_D^*$</td>
<td>$\Delta r_D^*$</td>
</tr>
<tr>
<td>$\Delta \mu$</td>
<td>+0.00111</td>
<td>+0.00227</td>
<td>+0.00433</td>
<td>+0.00529</td>
</tr>
<tr>
<td>$\Delta \sigma$</td>
<td>+0.00104</td>
<td>+0.00107</td>
<td>+0.00069</td>
<td>+0.00000</td>
</tr>
<tr>
<td>$\Delta \alpha$</td>
<td>-0.00003</td>
<td>-0.00004</td>
<td>-0.00013</td>
<td>-0.00021</td>
</tr>
<tr>
<td>$\Delta \lambda$</td>
<td>+0.00000</td>
<td>+0.00001</td>
<td>+0.00000</td>
<td>+0.00005</td>
</tr>
<tr>
<td>$\Delta \tau$</td>
<td>+0.00017</td>
<td>+0.00038</td>
<td>+0.00064</td>
<td>+0.00129</td>
</tr>
</tbody>
</table>
Table III
Optimal Input Prices with a Prepayment Penalty*

This table details the optimal time deposit rates with and without a prepayment penalty of 0.5%, using the following base case parameters:

\[
\mu = 0.0237, \sigma = 0.007, \alpha = 1.0, \tau = 0.10, \text{ and } \lambda = 0.06.
\]

<table>
<thead>
<tr>
<th></th>
<th>( T_M = 0.5 )</th>
<th>( T_M = 1.0 )</th>
<th>( T_M = 1.5 )</th>
<th>( T_M = 2.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Case (( \gamma = 0 ))</td>
<td>( r_D^* = 0.02687 )</td>
<td>( r_D^* = 0.03997 )</td>
<td>( r_D^* = 0.05087 )</td>
<td>( r_D^* = 0.05922 )</td>
</tr>
<tr>
<td>( \gamma = 0.005 )</td>
<td>( r_D^* = 0.02187 )</td>
<td>( r_D^* = 0.03497 )</td>
<td>( r_D^* = 0.04572 )</td>
<td>( r_D^* = 0.05383 )</td>
</tr>
</tbody>
</table>

* The typical commercial bank structures their early withdrawal penalties based on the original time to maturity of time deposits. For example, a CD that is issued with 12 months or less to maturity will have 1 to 3 months worth of interest as an early withdrawal penalty. A CD with 1 year or greater to maturity will have 3 to 6 months interest as a penalty. These penalties vary from bank to bank. Using these guidelines, a 2.0%, 1 year CD will have a 0.5% penalty on average. A 4%, 2 yr. CD will have between 0.5% and 1.0% penalty on average. The overall withdrawal penalty as a percentage of earned interest decreases the longer the customer waits for early withdrawal. Based on current market rates and average bank penalties, it is reasonable to choose \( \gamma = 0.005 \) (0.5%) for our simulations.
### Table IV: Relationship between interest rates and Real GDP

This table details the correlation between one year treasury interest rates and Real Gross Domestic Product during five time intervals of rising short term rates. The correlation coefficient and t-statistic for their significance are given for each interval.

<table>
<thead>
<tr>
<th>Date</th>
<th>Correlation coefficient (ρ)</th>
<th>t-statistic*</th>
</tr>
</thead>
<tbody>
<tr>
<td>8/22/86-10/16/87</td>
<td>.8634</td>
<td>3.423</td>
</tr>
<tr>
<td>2/12/88-1/27/89</td>
<td>.9384</td>
<td>4.704</td>
</tr>
<tr>
<td>10/8/93-5/13/94</td>
<td>.9476</td>
<td>4.195</td>
</tr>
<tr>
<td>6/10/94-12/30/94</td>
<td>.8658</td>
<td>1.730</td>
</tr>
<tr>
<td>10/23/98-5/19/00</td>
<td>.9613</td>
<td>9.869</td>
</tr>
</tbody>
</table>

*the computed value of \( t = \rho \cdot \sqrt{\frac{N-2}{1-\rho^2}} \), with N-2 degrees of freedom
Figure 1: Base Case with $T_m = 0.5$

This figure shows the expected cost of deposit funding on the vertical axis and the time deposit rate on the horizontal axis. $E(CDF)$ as a function of $r_D$ is plotted using the following base case parameters: $\mu = 0.0237, \sigma = 0.007, \tau = 0.10, r_0 = 0.0025, \lambda = 0.06, T_m = 0.50$. The optimal rate, the $r_D^*$ that minimizes the expected cost of deposit funding, is listed below the graph. The figure is plotted using Mathematica.

$r_D^* = 0.02687$
Figure 2: Base Case with $T_M = 1.0$

This figure shows the expected cost of deposit funding on the vertical axis and the time deposit rate on the horizontal axis. $E(CDF)$ as a function of $r_D$ is plotted using the following base case parameters:

$\mu = 0.0237, \sigma = 0.007, \tau = 0.10, r_0 = 0.0025, \lambda = 0.06, T_M = 1.0$. The optimal rate, the $r_D^*$ that minimizes the expected cost of deposit funding, is listed below the graph. The figure is plotted using Mathematica.

$E(CDF) = f(r_D)$

$r_D^* = 0.03997$
Figure 3: Base Case with $T_M = 1.5$

This figure shows the expected cost of deposit funding on the vertical axis and the time deposit rate on the horizontal axis. $E(CDF)$ as a function of $r_D$ is plotted using the following base case parameters:

$\mu = 0.0237, \sigma = 0.007, \tau = 0.10, r_0 = 0.0025, \lambda = 0.06, T_M = 1.50$. The optimal rate, the $r_D^*$ that minimizes the expected cost of deposit funding, is listed below the graph. The figure is plotted using Mathematica.

$r_D^* = 0.05087$
Figure 4: Base Case with $T_M = 2.0$

This figure shows the expected cost of deposit funding on the vertical axis and the time deposit rate on the horizontal axis. $E(CDF)$ as a function of $r_D$ is plotted using the following base case parameters:

$\mu = 0.0237, \sigma = 0.007, \tau = 0.10, r_0 = 0.0025, \lambda = 0.06, T_M = 2.0$. The optimal rate, the $r^*_D$ that minimizes the expected cost of deposit funding, is listed below the graph. The figure is plotted using Mathematica.

$r^*_D = .05922$