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# Feedback and the Success of Irrational Investors

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# Feedback and the Success of Irrational Investors

We provide a model in which irrational investors trade based upon considerations that are not inherently related to fundamentals. However, because trading activity affects market prices, and because of feedback from security prices to cash flows, the irrational trades influence underlying cash flows. As a result, irrational investors can, in some situations, earn positive expected profits. These expected profits are not market compensation for bearing risk, and can exceed the expected profits of rational informed investors. The trades of irrational investors can distort real investment choices and lower ex ante firm values, even though stocks prices follow a random walk.

# I Introduction

Investors often share common misconceptions, and participate in common errors of analysis. For example, a substantial number of investors employ technical rules that are supported by neither conceptual considerations nor empirical evidence. Fads in industry sectors, methods of security analysis, and simplistic theories of the stock market tend to proliferate through the mass media and word of mouth (Shiller (2000) discusses such phenomena). Investors who have fallen prey to common elementary errors, such as confusing the company Telecommunications Incorporated with the firm with ticker symbol TCI, have caused large price movements in one stock based upon news arrival in another unrelated stock (see Rashes (2001)). As another example, investors and prices sometimes react to the re-publication of information that is already public (see Ho and Michaely (1988) and Huberman and Regev (2001)).

Although interest in the stock market reached an all-time high during the 1990s, there is no indication that investor sophistication increased. Indeed, the tendency to value stocks based on ad hoc heuristics seemed to have increased. For example, it became popular to value tech firms based upon revenue rather than earnings; and to value ecommerce firms based upon eyeballs rather than revenue.[1] Many have alleged that these valuation methods were inappropriate, a criticism that, at least with the benefit of hindsight, seems to carry some weight. With the rise of the internet, there has been increased opportunity for investors to gain improved information about stocks. However, it is also easier for foolish stock market theories to be spread rapidly and

[1] See, for example, "Eyeballs, Bah! Figuring Dot-Coms' Real Worth," *Business Week Online*, October 30, 2000, or www.fvginternational.com/industries/industries_internet.html, that respectively discuss the invalidity and validity of these approaches.

widely.

There is a growing literature that explores the psychological basis for apparently irrational behavior and considers its effect on market prices.[2] While our paper contributes to this literature, our focus is very different. For example, in contrast to the existing literature, irrational trading in our model does not provide trading opportunities to those who are both rational and aware of the psychological biases of their less rational counterparts. Indeed, the irrational investors in our model earn positive expected profits that can exceed those of their more rational, informed, counterparts. From the perspective of the rational, but uninformed, investor, the market is informationally efficient. However, irrational trading, by affecting prices, causes shifts in the allocation of resources and lowers firm values on average.

To address these issues, our model includes four important elements:

- Feedback from stock prices to future cash flows. Feedback can arise for a variety of reasons. For example, a higher stock price may help firms attract customers and employees and may provides a cheap currency for making acquisitions.[3] The model developed in this paper considers a setting where feedback arises because higher stock prices encourage increased investment in complementary technologies.

- Irrational investors who are endowed with a common misguided way of thinking that rational investors cannot perfectly foresee. The inability of rational investors to foresee the trading of irrational investors may arise because there are potentially many irrational heuristics, and it is difficult, ex ante, to know which ones

[2] Hirshleifer (2001) reviews this literature.

[3] Subrahmanyam and Titman (2001) discuss feedback in a related context.

the irrational investors will find appealing. Alternatively, irrational investors may have developed a special adeptness at interpreting what would otherwise be irrelevant information. For example, these investors may have taken special courses at some earlier date on the evaluation of solar flares, or on meaningless technical indicators (e.g., on identifying genetic algorithms) and thereby developed a skill that the rational investors, who did not choose to take the course, do not have.

- Third, although irrational investors are identical ex ante, we assume that they do not complete their irrational analyses of the stock in question at the same time.

- Finally, there exist some rational investors with information about future cash flows. This final element is important because rational market makers, who will ignore the order flow if there are no informed investors, set prices.

When the above conditions are satisfied, the irrational investors who act early earn positive expected profits and the irrational investors who act late earn negative expected profits. This by itself is not particularly surprising. However, the ex ante expected profits of an irrational investor (before it is known whether he will detect the information early or late) are positive and under some conditions, even exceed the gains of the *informed* rational investors.

The irrational investors in our model are able to earn expected profits because the price response generated from irrational trades affects future cash flows, and hence, the future fundamental value of the stock. If this were not the case, then the increase in price that occurs when the irrational investors purchase the stock would on average be reversed when they subsequently sell, leading to negative overall profits. Instead, owing to feedback, the trades of early irrational investors in effect exploit information about the future order flow of investors with similar psychological biases. The early irrational investors make money not only because the later irrational investors subsequently drive prices too high, but also because the later investors affect fundamentals. This fundamental-based component of the capital gain increases the average profits of both the early traders and the later traders, so that the combined profits are on average positive.

The above argument does *not* require that the irrational investors be sophisticated enough to anticipate the feedback effect. Indeed, we assume that irrational traders ignore the reverse causality from prices to cash flows. The sequential arrival of correlated irrational trades automatically positions the early irrational investors to profit, regardless of whether they understand either the feedback or the sequential structure.

Furthermore, even though irrational trading affects prices, the model does not necessarily imply that an uninformed researcher will detect evidence of market inefficiency. In particular, there is no trading rule that can earn abnormal profits based upon publicly available information. When rational observers see a price move, they face an inference problem of disentangling whether it resulted from intertemporally correlated irrational trades, or from uncorrelated noise trades. From the perspective of the uninformed observer, the market is efficient and prices follow a random walk.

However, irrational trading affects prices, and in turn, real investment choices. This is especially the case in industries where feedback is important and where private information has a high ex ante variance. Firms in these industries may succeed or fail as a consequence of irrational but self-reinforcing beliefs of the irrational investors. We conclude that ex ante, irrational trading diminishes value. Specifically, when the variability of irrational beliefs increases, resources are allocated less efficiently and ex ante firm value declines.

Our analysis is related to a number of papers in the literature. First, the issue of feedback from the stock market to cash flows was analyzed by Subrahmanyam and Titman (2001) in a setting with fully rational investors. Here we examine the consequences of imperfectly rational trading. The relevance of having investors receive information at different times was previously explored by Froot, Scharfstein, and Stein (1992) and Hirshleifer, Subrahmanyam, and Titman (1994), both of which considered settings with rational investors and no feedback.[4] There have also been other papers that have considered very different settings in which irrational traders earn higher expected profits than fully rational ones. In DeLong, Shleifer, Summers, and Waldmann (1991), overconfident investors with fundamental information underestimate risk, and therefore take larger long positions in risky assets. Therefore these overconfident investors earn higher returns than their rational counterparts.[5] Thus, in DeLong et al, high irrational returns reflect a premium for market risk. In contrast, we consider a setting where prices are set by risk neutral marketmakers, so that the expected profits of irrational investors in our model are *not* a market compensation for bearing risk. Several other papers have

[4]In the former paper, early irrational traders as well as early informed traders can make money by initially pushing up stock prices and then reversing their trades at a price that is the same as that obtained by the late-arriving traders. This is because all of the traders are constrained to reverse their positions immediately following the late irrational trades. In Hirshleifer, Subrahmanyam and Titman (1994), this reversal is endogenized by assuming that early informed investors are risk averse, and therefore wish to unload their risky positions when the late-informed arrive and move prices in the direction of their information. Irrational trading is not profitable in their framework, because there is no feedback.

[5]Similarly, in Blume and Easley (1990) and Kogan, Ross, Wang, and Westerfield (2002), irrational investors have utility functions that are closer to logarithmic than those of rational investors, which leads them to accumulate more wealth in the long-run than their rational counterparts.

examined how in an imperfectly competitive securities market overconfident informed traders can benefit by intimidating other informed traders.[6] All of these papers, however, require that irrational investors have direct information about fundamentals. In contrast, in our paper irrational traders earn greater expected profits than informed investors even though they have no inherent information about fundamentals.

The remainder of the paper is structured as follows. Section II describes the economic setting. Section III derives asset demands, expected profits to different kinds of traders, and numerical comparisons of the profitability of different kinds of traders. Section IV provides a simplified version of the model and analytic results about the profitability of different trader classes, while Section V concludes.

# II The Economic Setting

## II.1 Firms

Consider a setting in which a firm has an investment opportunity with a random payoff. Claims on this opportunity are traded at each of dates 1 and 2, at prices $P_1$ and $P_2$. The claim pays off the amount $F = \theta + \epsilon + \delta$ at date 3, where $\theta$ and $\epsilon$ are independent normal random variables with mean zero. As described below, the variable $\delta$ is determined by the stock prices realized at dates 1 and 2.

There are a number of ways that feedback of this type can be modeled. For example, in Subrahmanyam and Titman (2001), feedback arises because outside stakeholders

[6] See Kyle and Wang (1997), Wang (1998) and Fischer and Verrecchia (1999). In a competitive securities market, Hirshleifer and Luo (2001) show that overconfident investors who trade aggressively in response to their private information signals can exploit liquidity traders more profitably than rational investors.

of a firm, like its workers, suppliers and customers, are more willing to do business with the firm if it has a rising stock price and is perceived to be a winner. While a stakeholder framework can easily be adapted to our setting and delivers essentially equivalent results, it does not capture the possibility that stock prices can influence real investment, which is captured by our alternative approach.

We capture feedback by assuming that there are two firms that sell interdependent products. Examples could include, among others, computer hardware and software firms, media companies and consumer electronics companies, and airlines and aircraft companies. Within this setting, an increase in Firm 1's stock price provides a signal to Firm 2 that demand for Firm 1's product, and hence its own product, has increased. In response to this signal Firm 2 increases its investment, allowing it to increase the production of its complementary product, which in turn, increases Firm 1's value.[7]

As described above, the interaction between firms is a network externality, but not a pure consumption externality, such as the benefit to email users of having other people who read email. Rather, it involves spillovers between the productive decisions of one firm and the demand for the product of another firm, as with the benefit provided to PC sellers of having other companies develop useful software for PCs. However, similar feedback would obtain in an alternative setting in which there are pure consumption network externalities.

This setting captures, in a very simple way, what some refer to as network exter-

[7] In a more complex setting, an increase in Firm 2's investment will in turn lead to further investment by Firm 1, leading to a closed loop of interactions. Such interdependence of investment and production choices by firms with complementary products was previously examined by Scitovsky (1954) and Shleifer (1986). For tractability in analyzing information and securities market equilibrium, we do not consider this extension, which would lead to price changes that are not normally distributed.

nalities. More generally, as more firms invest in complementary technologies, and as more individuals consume the products of these technologies, the value of the technologies increases. For example, a more complex setting could describe a situation where a software producer's stock price influences the decisions of the suppliers of ancillary products as well as its customers in ways that feed back to the software producer's future cash flows.

To further simplify the model we assume that the first firm, which is publicly traded, has assets that generate income but has no additional investment opportunities. The second firm, which we assume is privately owned by investors who do not trade the shares of the first firm, has an investment opportunity whose expected value is determined in part by economic conditions that are reflected in the value of the first firm. Since this investment increases the production of Firm 2's complementary product, the investment increases Firm 1's value.[8] We assume that the firms are not able to write binding contracts on Firm 2's output, and that there are transactions costs that prohibit the purchase of one firm by the other.

We assume that the payoff on the second firm's growth opportunity is positively correlated with the $\theta$ component of the payoff on the first firm's claim. Specifically, we assume that the growth opportunity payoff is given by

$$G = \bar{G} + \theta - 0.5C^2 \tag{1}$$

in period 3, where $\bar{G}$ is a positive constant and $C$ is the amount of capital devoted to the project. For convenience, we assume that the second firm is funded by a private

[8] One can derive most of our results in an alternative setting with one firm and employees who increase their effort when the firm's prospects, as conveyed by stock prices, increase. However, in such a setting we would not be able to examine distortions in physical investment.

financier or by investors that do not trade claims in firm 1. We further assume that each unit of capital invested in firm 2 increases the payoff on the first firm's claim by $k$ units, so that

$$\delta = kC.$$

Firm 2 acts to maximize the expected value of its growth opportunity at time 2. Maximizing the date 2 conditional expectation of (1) with respect to $C$, we obtain

$$C = E(\theta | P_1, P_2). \tag{2}$$

In equilibrium, this rational expectation is an increasing function of the date 1 and date 2 prices, so that there is positive feedback from the stock price of firm 1 to the amount of investment by firm 2.[9]

The date 3 value of firm 1's investment opportunity is

$$F = \theta + \epsilon + kE(\theta | P_1, P_2). \tag{3}$$

Thus, the higher the rational conditional expectation of $\theta$, the higher is firm 1's payoff at date 3. Since in equilibrium this expectation is increasing in prices, there is positive feedback from firm 1's stock price to its expected future cash flow.

Substituting for the optimal $C$ from (2) into (1), the ex ante expected value of firm 2's growth opportunity is

$$G = \bar{G} + 0.5E\left[E(\theta | P_1, P_2)\right]^2 = \bar{G} + 0.5\text{var}\left[E(\theta | P_1, P_2)\right]. \tag{4}$$

[9]Our two-firm modeling structure provides a relatively simple setting in which the first firm's *actual* cash flow increases *linearly* as a function of its *perceived* cash flow as inferred from its stock price. Feedback from investor perceptions to corporate investment could similarly be captured within a setting with a single firm that can realize greater growth opportunities when its stock price is higher. Again, however, the non-linear relation between stock prices and cash flows in such a setting leads to loss of analytical tractability.

Since the primary focus of our analysis is on the market for firm 1's claim, for the remainder of the paper we refer to firm 1 as "the firm," as distinct from firm 2.

## II.2 The Investors

We assume that there are two types of rational informed investors. The *early informed* learn precisely the realization of $\theta$ when the market opens at date 1, while the *late informed* do not receive any information at date 1, but learn the realization of $\theta$ when the market opens at date 2. The error term $\epsilon$ remains unknown at both trading dates.

We assume that all traders behave competitively. The mass (or measure) of the early informed traders is denoted by $M$, while the total mass of early- and late-trading investors is normalized to unity, so that the mass of late-trading investors is $1 - M$. Both groups of traders have negative exponential utility over terminal wealth with a common absolute risk aversion coefficient $R$.

In addition, there is a group of utility-maximizing irrational traders who mistakenly believe that the security pays off $\eta + \epsilon$, where $\eta$ is a random variable that is independent of all other exogenous random variables and is also normally distributed with mean zero. Thus, $\eta$ has no inherent relation to fundamentals, and endogenously becomes related only through feedback from the firm's stock price to cash flows. The irrational traders observe the realization of $\eta$, but the rational traders do not. There is a mass $N$ of early irrational traders and a mass $1 - N$ of late irrational traders. For simplicity, we assume that the irrational traders do not anticipate the feedback effect.

Liquidity demand shocks for the claim in amounts of $z_1$ and $z_2$ arrive at dates 1 and 2 respectively. These shocks are normally distributed with mean zero, and are independent of each other and of $\theta$ and $\epsilon$.

There is also a group of risk neutral market makers, who possess no information about the fundamental value of the risky security. These agents represent a competitive fringe of risk neutral traders (e.g., floor brokers, scalpers, or institutions who monitor trading floor activities) who are willing to absorb the net demands of the other traders at competitive prices.

Our equilibrium concept closely parallels that used by Vives (1995) and Hirshleifer, Subrahmanyam, and Titman (1994). We assume that at both dates 1 and 2, informed investors submit demand schedules ('limit orders') as a function of their information and the market prices. The risk neutral market makers observe the combined demand schedules of the informed and liquidity traders and set competitive prices at each date. Let $\gamma_1, \gamma_2$ denote the aggregate demand schedules at dates $t = 1, 2$.

Because market makers are risk neutral and competitive, they set prices that are semi-strong form efficient. Thus, at each date the security's price is equal to the expectation of the terminal cash flow of the security, conditional on the information set of the market makers, i.e.,

$$\begin{aligned} P_1 &= E\left[F|\gamma_1(\cdot)\right] \\ P_2 &= E\left[F|\gamma_1(\cdot), \gamma_2(\cdot)\right]. \end{aligned} \tag{5}$$

The date 3 price, $P_3$, is equal to the final value of the claim, $F$. We will consider linear equilibria, wherein pricing functions are linear in the random variables $\theta$, $\eta$, $z_1$, and $z_2$. Given such functions, it can easily be shown that the demand schedules can be written as

$$\begin{aligned}\gamma_1(P_1) &= \tau_1 + f(P_1) \\ \gamma_2(P_2) &= \tau_2 + g(P_2),\end{aligned} \tag{6}$$

where $f(\cdot)$ and $g(\cdot)$ are linear functions, and $\tau_1$ and $\tau_2$ are linear combinations of the informational variable $\theta$, the irrational noise variable $\eta$, and the liquidity trades $z_1$ and $z_2$. The informative parts of the demand schedules are the variables $\tau_1$ and $\tau_2$. We therefore have

$$\begin{aligned}P_1 &= E\left[F|D_1(\cdot)\right] = E\left[F|\tau_1\right] \\ P_2 &= E\left[F|D_1(\cdot), D_2(\cdot)\right] = E\left[F|\tau_1, \tau_2\right].\end{aligned} \tag{7}$$

# III Equilibrium

## III.1 The Demands of Investors

To derive the linear equilibria, we begin by postulating that the prices are linear functions of the private information variable $\theta$ and the liquidity demand shocks to date, such that

$$P_1 = a_1\theta + a_2\eta + a_3 z_1 \tag{8}$$

$$P_2 = b_1\theta + b_2\eta + b_3 z_1 + b_4 z_2. \tag{9}$$

Let $x_1$ and $x_2$ represent the demands of the rational early informed agents at date $t$. Since the date 2 wealth is conditionally normally distributed, one can use the mean-variance framework and standard methodology to show that the optimal risky holdings of each early- and late-trading individual at the end of date 2 are identical and are

$$x_2 = \frac{\theta + kE(\theta|P_1, P_2) - P_2}{Rv_\epsilon}. \tag{10}$$

Let $E_r(P_2)$ and $v_r(P_2)$ denote the mean and variance of $P_2$ conditional on the information set of the early informed at date 1. This information set consists of $\theta$ and the market price $P_1$, so although the rational informed traders do not know $\eta$ precisely, they infer it partially from market prices.[10] The appendix shows that the optimal date 1 demand of an early-informed trader is

$$x_1 = \left[\frac{E_r(P_2) - P_1}{R}\right]\left[\frac{1}{v_r(P_2)} + \frac{1}{v_\epsilon}\right] + \frac{\theta - E_r(P_2)/(1+k)}{Rv_\epsilon}. \tag{11}$$

The demand represented by (11) consists of two components, one to exploit the expected price appreciation across dates 1 and 2, and another to lock in at the current price the expected demand at date 2.

It can easily be shown that the date 1 demand of the late-trading informed investors equals zero in equilibrium. Intuitively, this occurs for two reasons. First, the equilibrium date 1 price does not offer a risk premium because of the presence of risk-neutral market makers. Second, the late-trading investors cannot hedge their date 2 demand in advance, because conditional on their date 1 information set (which does not contain the informational variable $\theta$), the expected date 2 price is unbiased, so that their expected date 2 trade is zero.[11]

Let $y_1$ and $y_2$ represent the demands of the early irrational traders at dates 1 and

[10]The next section discusses a simplified case in which the informed are allowed to precisely observe the variable $\eta$ with a lag. Similar results obtain in this alternative formulation.

[11]The proof of this intuitive assertion is available from the authors upon request.

2. Again the date 2 demands of the early and late irrational traders are identical,

$$y_2 = \frac{\eta - P_2}{Rv_\epsilon}. \tag{12}$$

Let $E_n(P_2)$ and $v_n(P_2)$ denote the mean and variance of $P_2$ conditional on the information set of the early irrational traders at date 1 (this information set includes $P_1$ and their signal $\eta$). Analogous to (11), the date 1 demand of an early irrational trader is

$$y_1 = \left[\frac{E_n(P_2) - P_1}{R}\right]\left[\frac{1}{v_n(P_2)} + \frac{1}{\sigma_\epsilon^2}\right] + \frac{\eta - E_n(P_2)}{Rv_\epsilon}, \tag{13}$$

whereas the date 1 demand of a late irrational trader equals zero for the same reason as for the late informed trader.[12]

## III.2 Equilibrium Prices

Given the aggregate demands $\gamma_1 = Mx_1 + Ny_1 + z_1$ and $\gamma_2 = x_2 + y_2 + z_1 + z_2$, it can be shown that the components of the demand schedules which are informative about final value take the following form:

$$\tau_2 = \theta + \eta + Rv_\epsilon(z_1 + z_2),$$

$$\tau_1 = M\left\{\left[\frac{E_r(P_2)}{R}\right]\left[\frac{1}{v_r(P_2)} + \frac{k}{(1+k)v_\epsilon}\right] + \frac{\theta}{Rv_\epsilon}\right\} + N\left[\frac{E_n(P_2)}{Rv_n(P_2)} + \frac{\eta}{Rv_\epsilon}\right] + z_1.$$

Given the expressions for the early and late irrational trader demands, it is easy to show that the expected profits of the late irrational traders are

$$\pi_{nl} = E[x_2(F - P_2)] = v_\theta\left[\frac{-b_1}{Rv_\epsilon}\left(1 - \frac{b_1}{1+k}\right)\right] + v_\eta\left[\frac{1-b_2}{Rv_\epsilon}\left(-\frac{b_2}{1+k}\right)\right] + v_z\frac{(b_3^2 + b_4^2)}{(1+k)Rv_\epsilon},$$

and the expected profits of the early irrational traders are

$$\pi_{ne} = E[x_2F - (x_2 - x_1)P_2 - x_1P_1] = R^{-1}v_\theta\left\{\frac{n_1}{v_n(P_2)} - a_1\left[\frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon}\right]\right\}(b_1 - a_1)$$

[12] A proof is available upon request.

$$+R^{-1}v_\eta \left\{ \frac{n_2}{v_n(P_2)} - a_2 \left[ \frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon} \right] \right\} (b_2 - a_2)$$

$$+R^{-1}v_z \left\{ \frac{n_3}{v_n(P_2)} - a_3 \left[ \frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon} \right] \right\} (b_3 - a_3) + \pi_{nl}.$$

The above expressions indicate that the early irrational traders profit from feedback because the date 2 price is correlated both with their noise $\eta$ as well as with the fundamental $\eta$. Indeed, the stronger the feedback, the more the price moves at date 2.

The solution process for the equilibrium proceeds as follows. First, observe that $E_r(P_2) = E(P_2|P_1, \theta)$, and can be written as $r_1\theta + r_2\eta + r_3 z_1$. Further, $E_n(P_2) = E(P_2|P_1, \eta)$ and can be written as $n_1\theta + n_2\eta + n_3 z_1$. In addition, $v_r(P_2)$ and $v_n(P_2)$ are not functions of the realizations of the random variables, but are well-known functions of the price coefficients and variances of the random variables.[13] These facts allow us to solve for the equilibrium value of coefficients $a_t$ and $b_t$, $t = 1, 2, 3$ in the price functions postulated in (9) and (8); details appear in the appendix. The following proposition describes properties of equilibrium price changes and the equilibrium expected profits of the rational and irrational traders.

**Proposition 1** *In the general model:*

1. *Price changes are serially uncorrelated.*

2. *There exists a non-empty set of exogenous parameter values under which the ex ante expected profits of the irrational traders are positive and exceed those of the*

[13]We assume that irrational traders accurately understand the coefficients of the market makers' pricing function. Our purpose is not realism, but modeling parsimony: we wish to show that a single irrationality can, when combined with feedback, generate excess profits for irrational traders. The assumption could also reflect learning by irrational investors, either through experience or communication, about the pricing function. Alternatively, the irrational investors may understand that other investors think they are irrational, so the irrational investors understand that the market makers are not going to treat the irrational trades as if they were informed.

*rational informed traders.*

Part 1 follows from the fact that market makers are risk-neutral and set prices to be expectations of final value conditional on all public information. Hence prices follow a martingale. Part 2 of the proposition is demonstrated by means of numerical comparative statics on irrational trader profits with respect to changes in the feedback parameter $k$ (See Figure 1), and on the expected profit differential between irrational and rational traders (See Figure 2).

In Figure 1, the late irrational expected profits are always negative. The early irrational expected profits are negative when $k = 0$, but increase as $k$ increases. The ex ante total expected profits also start negative and increase with $k$.

Further numerical analysis (not reported) indicates that the threshold level of $k$ above which the total irrational expected profits are positive is increasing in the ratio $v_\theta/v_\eta$, the ratio of the variances of the inherent information,$\theta$, and of the irrational belief, $\eta$, and decreasing in the variance of liquidity trade $v_z$; these are intuitive results. Increasing $v_\theta/v_\eta$ increases the strength of the price move against the late irrational traders. This adversely affects their expected profits, and consequently increases the threshold $k$. In contrast, increasing $v_z$, the variance of liquidity trading causes the market to be more liquid, which increases ex ante irrational trader profits.

We also find, somewhat surprisingly, that the threshold level of $k$ is decreasing in the risk $v_\epsilon$ that is not resolved prior to the terminal date by the signal received by the rational informed. This occurs because increasing $v_\epsilon$ decreases the size of the position held by the late irrational traders, which mitigates their losses. Consequently, weaker feedback suffices for the ex ante expected profits of the irrational traders to be positive.

Figure 2 compares the total ex ante expected profits earned by irrational traders with the total ex ante expected profits earned by rational informed traders as a function of the feedback parameter. We graph the difference in profits relative to the informed profit (which is always positive) in the denominator. When the feedback parameter $k$ is low, this difference is negative, indicating that irrational traders do worse than rational informed traders. However, irrational traders do better in relative terms as $k$ increases, and earn higher profits than rational informed traders when $k$ is greater than approximately 12.

## III.3 Discussion

The findings in Figures 1 and 2 confirm the intuition offered in the introduction. When early irrational traders buy, market makers cannot be sure whether the trading arises from the intertemporally correlated trades of the irrational traders or the informed traders. So price is driven upward, which through feedback increases the firm's fundamental value. The arrival of later individuals who make similar irrational errors drives the price and, owing to feedback, fundamentals up further. The later buy orders cannot be entirely anticipated by the market-maker, because there was a chance that the early buys came from intertemporally uncorrelated liquidity traders rather than intertemporally correlated irrational traders.

The late irrational traders on average lose money because they overpay even relative to the improved fundamental. However, from an ex ante perspective, the expected profits of an irrational trader (who could turn out to trade either early or late) are positive. These irrational traders profit, when they trade early, by effectively trading on 'inside information' about the future order flow and, through feedback, its effect on fundamentals. If feedback is strong enough, this "pseudo-information" can be more valuable than the actual information acquired by the informed traders.

Irrational trading, in this model, affects real investment as well as prices. Indeed, when irrational sentiment is positive, more resources are committed to the growth opportunity, affecting the firm's cash flow prospects.[14] Thus, in our model irrational trading influences firm values. Nevertheless, in contrast to other models of irrational trading, the market is efficient, in our model, in the sense that an uninformed investor cannot earn a positive profits.[15]

# IV A Simplified Model

The general model provides intuitive numerical comparative statics but does not lend itself to analytical results. To develop more insights using closed-form solutions to the equilibrium, and to verify the robustness of our conclusions, we consider a simplified version of the model in which each early irrational investor trades at date 1, and reverses his trade at date 2. Further, he believes that the end of period payoff is $\eta + \epsilon$.[16] Despite the fact that this assumption requires the irrational trader to be suboptimally myopic (from his standpoint), we will show that irrationality can be ex ante profitable.

[14]These conclusions follow from our verification that in our numerical example, both the amount of capital committed to firm 2's growth opportunity and the value of firm 1's claim are positively correlated with the pre-trade irrational belief, $\eta$. In Section IV, we derive similar conclusions analytically within a simplified framework.

[15]Similarly, in many standard market microstructure models, prices are efficient, but if liquidity or "noise" trading is viewed as irrational, then irrational trading affects prices.

[16]This assumption can be justified by assuming that he trades at date 1 expecting $\eta$ to be publicly revealed at date 2, but it is not, so he is convinced that he made a mistake and hence reverses his trade.

Within the simplified model, the total mass of early irrational traders is specified to equal $M$. We assume that there is no informed trading at date 1, but at date 2 a unit mass of informed agents enter the market. In addition a mass $1 - M$ of late irrational traders enter the market at date 2. Under these assumptions, it is easy to show that the irrational investor's trade equals

$$y_1 = \frac{\eta - P_1}{Rv_\epsilon}.$$

The date 2 trades of the rational and irrational informed agents as a function of the price remain unchanged in this framework.

The market makers observe the variables $\tau_1 \equiv M\eta + Rv_\epsilon z_1$ at date 1 and $\tau_2 \equiv \theta + (1-M)\eta + Rv_\epsilon(z_1 + z_2)$. The coefficients on the date 1 price (on $\eta$ and $z_1$) are zero (market makers do not learn anything from the date 1 information set that they will not learn from the date 2 information set). The date 2 price is given by $P_2 = E(F|\tau_1, \tau_2) = (1+k)E(\theta|\tau_1, \tau_2)$. Let $E(\theta|\tau_1, \tau_2)$ be $b_1'\theta + b_2'\eta + b_3'z_1 + b_4'z_2$. Then, using standard properties of normal distributions, the equilibrium values of the coefficients in this linear function are given as follows:

$$b_1' = v_\theta(M^2 v_\eta + R^2 v_\epsilon^2 v_z)/D \tag{14}$$

$$b_2' = R^2 v_\epsilon^2 v_\theta v_z (1 - 2M)/D \tag{15}$$

$$b_3' = -MRv_\epsilon v_\eta v_\theta (1 - 2M)/D \tag{16}$$

$$b_4' = Rv_\epsilon v_\theta (M^2 v_\eta + R^2 v_\epsilon^2 v_z)/D, \tag{17}$$

where

$$D \equiv M^2 v_\eta (5R^2 v_\epsilon^2 v_z + v_\theta) - 4MR^2 v_\epsilon^2 v_\eta v_z + R^2 v_\epsilon^2 v_z (R^2 v_\epsilon^2 v_z + v_\eta + v_\theta) > 0. \tag{18}$$

It is intuitively clear that the early irrational traders cannot profit unless the mass of late irrational traders is sufficiently large. Within the context of our model, it turns out that the expected profits of the early irrational traders are positive if and only if $M < 0.5$. For the remaining part of this section, we will explore in more detail the implications that arise when irrational traders earn expected profits, and will thus assume that the condition $0 < M < 0.5$ holds. Given this assumption, and the equilibrium price coefficients described above, it is straightforward to derive the following proposition.[17]

**Proposition 2** *1. The irrational signal $\eta$, and the trades of the early irrational traders are both positively correlated with the fundamental value of the firm's claim so long as $k > 0$.*

*2. The correlation between the trades of the late irrational traders and the firm's date 3 fundamental value is of ambiguous sign. However, for $M$ close to zero, this correlation is positive so long as $k > 0$ and $v_\eta$ is sufficiently large.*

Proposition 2 indicates that irrational trading affects the firm's fundamental value. Indeed, the trades of the early irrational investors are positively correlated with the fundamental value so long as there is feedback from prices to cash flows. For small enough mass of the early irrational $M$, the correlation between the trades of the late irrational investors and the fundamental value is positive so long as the variance of the irrational signal is sufficiently large. This is intuitive; if there are not too many early irrational investors, then late irrational trades are of the same sign as those of the early irrational traders. So both are positively correlated with fundamentals.

[17] All propositions in this section are proved in the appendix.

Since irrational trading feeds back to fundamentals, in equilibrium firms may succeed or fail as a result of self-reinforcing irrational beliefs. Our analysis suggests that firms in an industry about which irrational investors are unduly optimistic appreciate in value because of feedback, partly confirming that optimism. On the other hand, irrational pessimism about an industry causes stocks in that industry to fall, partially confirming the pessimism. This feedback effect is especially important in industries with complementary product network externalities and high levels of stakeholder investment.

The following proposition describes the correlation between irrational trades and price moves.

**Proposition 3** *1. The trades of the early irrational traders are positively correlated with the date 2 price move,* $P_2 - P_1$.

*2. The trades of the late irrational traders are positively correlated with the date 2 price move,* $P_2 - P_1$ *if and only if*

$$v_\theta(1+k)(R^2 v_\epsilon^2 v_z + M^2 v_\eta) < R^2 v_\epsilon^2 v_\eta v_z (1-2M). \quad (19)$$

*3. The trades of the late irrational traders are positively correlated with the trades of the early irrational traders if and only if* $1+k > b_2'^{-1}$, *i.e., if and only if*

$$\frac{D}{R^2(1+k)v_\epsilon^2 v_\theta v_z(1-2M)} > 1. \quad (20)$$

Part 1 indicates that irrational traders do indeed affect prices. Intuitively, market makers thinks that their trades might have come from informed investors. Parts 2 and 3 provide conditions under which the late irrational investors tend to trade in the same direction as early irrational investors, and in the same direction as the date 2 price move.

It is possible for the late irrational trades to be negatively correlated with both the early irrational traders and the date 2 price move. This happens when the inequalities in (19) and (20) are reversed. To see why, suppose that there is strong feedback, so that the rational informed buy very aggressively on a positive signal. In this case, late irrational investors may short stock even if their signal is positive because the price they face is high when there is strong informed buying.

We now describe autocorrelation patterns in prices and order flows.

**Proposition 4** *1. Equilibrium order flows are positively autocorrelated.*

*2. Unconditional price changes are serially uncorrelated in equilibrium; i.e.,* $\text{cov}(P_3 - P_2, P_2 - P_1) = 0$.

*3. Equilibrium prices exhibit positive autocorrelation conditional on the irrational signal, and negative autocorrelation conditional on the rational signal. Specifically,*

$$\text{cov}(P_3 - P_2, P_2 - P_1 | \eta) > 0$$

$$\text{cov}(P_3 - P_2, P_2 - P_1 | \theta) < 0.$$

Part 1 indicates that order flows are serially dependent because of the sequential arrival of irrational traders. Part 2 again confirms that the results are not driven by market inefficiency; prices at each date are equal to the expected value of an equity claim on the firm conditional on all publicly available information. Parts 1 and 2 show that serial dependence in order flow is not inconsistent with serial independence in price movements. These results are thus consistent with the positive autocorrelation in order imbalances but virtually zero autocorrelation in daily stock returns documented by Chordia, Roll, and Subrahmanyam (2002).

Part 3 of the above proposition documents that prices exhibit persistence after controlling for the irrational signal. This happens because order flows are noisy transformations of the rational investors' trades, so that prices underreact to the valid information signal $\theta$. On the other hand, because the trades of the irrational investors get mixed in with those of the informed traders, prices overreact to irrational trades and consequently exhibit reversals after controlling for the rational signal.

The expected profits of the early irrational traders in terms of the price coefficients are

$$\pi_{ne} = E[x_1(P_2 - P_1)] = \frac{(1+k)b_2' v_\eta}{R v_\epsilon},$$

and those of the late irrational traders are

$$\pi_{nl} = E[x_2(F - P_2)]$$

$$= \frac{v_\theta[-b_1'(1+k)](1-b_1') + v_\eta[1 - b_2'(1+k)](-b_2') + (1+k)v_z(b_3'^2 + b_4'^2)}{R v_\epsilon}.$$

The ex ante expected profits of the irrational traders are $\pi = M\pi_{ne} + (1-M)\pi_{nl}$. Substituting for the price coefficients, we find that the ex ante expected profits are

$$\pi = \frac{R v_\epsilon v_\eta v_\theta v_z (1-2M)[(k+2)M - 1]}{D}, \tag{21}$$

where $D$ is as given in (18). This leads to the following proposition.

**Proposition 5** *1. If there is no feedback, i.e., $k = 0$, then the ex ante expected profits of irrational traders are always negative.*

*2. The ex ante expected profits of irrational traders are positive so long as $k > (1-2M)/M$.*

Part 1 confirms that when there is no feedback, irrational trading is unprofitable. Part 2 indicates that, in contrast, sufficient feedback makes irrational trading profitable. The intuition is the same as that outlined in the preceding section. Again, the expected profits are not a consequence of sophisticated exploitation of the feedback effect; indeed irrational investors are naïve and simply ignore this effect. They inadvertently profit from feedback because their trades are correlated with those of later irrational investors. The condition in part 2, under which the expected profits of the irrational traders are positive, emphasizes the role of early irrational traders. In particular, as the mass of early irrational traders, $M$, goes to zero, the bound on $k$ goes to infinity, thus indicating that for any finite level of feedback, a strictly positive mass of early irrational traders is necessary for the irrational traders to earn positive expected profits.

Further, the expected profits of the informed traders in terms of the coefficients $b'_i$ are

$$v_\theta(1-b'_1)^2 + v_\eta b_2'^2 + v_z(b_3'^2 + b_4'^2).$$

Again, substituting for the price coefficients, we find that expected informed profits, denoted by $\pi_i$ are

$$\pi_i = \frac{R^2 v_\epsilon^2 v_\theta v_z [v_\eta\{M^2 + (2M-1)^2\} + R^2 v_\epsilon^2 v_z]}{D}. \tag{22}$$

Comparing equations (21) and (22), we have the following proposition.

**Proposition 6** *A sufficient condition for the ex ante expected profits of irrational traders to be greater than the ex ante expected profits of informed traders is that*

$$k > \frac{R v_\epsilon \{v_\eta[(2M-1)^2 + M^2] + R^2 v_\epsilon^2 v_z\} + v_\eta(2M-1)^2}{M v_\eta(1-2M)}. \tag{23}$$

Proposition 6 indicates that, consistent with the numerical results of the general model, a sufficiently high feedback parameter causes the expected profitability of irrational traders to be greater than that of rational informed traders.

We next describe comparative statics on the expected profit differential between the irrational and the rationally informed traders.

**Proposition 7** 1. *The ex ante expected profit differential between irrational traders and informed traders is increasing in the feedback parameter, $k$.*

2. *The ex ante expected profit differential between irrational traders and informed traders is increasing in the variance of the signal observed by irrational traders, $v_\eta$, so long as*

$$k > \frac{[R^2 v_\epsilon^2 v_z + v_\theta(R v_\epsilon + 1)][1 - 2M]}{M(R^2 v_\epsilon^2 v_z + v_\theta)}. \tag{24}$$

3. *The ex ante expected profit differential between irrational traders and informed traders is increasing in the variance of the signal observed by informed traders, $v_\theta$, so long as*

$$k > \frac{v_\eta[(2M-1)^2(R v_\epsilon + 1) + R M^2 v_\eta v_\epsilon] + R^3 v_\epsilon^3 v_z}{M v_\eta(1 - 2M)}. \tag{25}$$

Proposition 7 indicates, consistent with intuition, that the performance of the irrational traders relative to the rational informed ones increases with the feedback parameter. Furthermore, the expected profits of the irrational traders relative to those of informed traders increase in the variance of the irrational signal. Intuitively, the coordinating signal and feedback drive trades and profits. Finally, the expected profit differential increases in the variance of information $v_\theta$ so long as the feedback effect is strong.

This is because as the ex ante variance of private information increases, the signal to noise ratio in the net demand increases, which strengthens the feedback resulting from irrational trades.

Our final proposition describes how irrational trading affects corporate resource allocation.

**Proposition 8** *1. The amount of capital allocated to the growth opportunity is positively correlated with the irrational signal $\eta$.*

*2. The ex ante volatility of capital allocated to the growth opportunity is increasing in the variance of the irrational signal, $v_\eta$.*

*3. The ex ante expected value of the growth opportunity is decreasing in the variance of the irrational signal, $v_\eta$.*

The above proposition describes how irrationality affects capital investment. Positive irrational sentiment, on average, increases corporate investment and vice versa. However, ex ante, irrational trading causes less informative prices and consequently poorer resource allocation. Thus, irrational investors on average decrease the efficiency of investment.

# V Concluding Remarks

In discussions of the efficient market hypothesis, financial economists often touch upon two points. The first point, which explains why we expect the hypothesis to hold at least approximately, is that investors who trade irrationally will lose money and thus, in the long run, will not have much influence on stock prices. The second point, which relates to why the hypothesis is important, is that informational efficiency of securities markets guides firms and investors to efficient allocations of capital and labor.

This paper addresses both issues. First, we show that when feedback from stock prices to cash flows is sufficiently strong, irrational investors can realize positive expected profits that exceed the expected profits of investors with fundamental information. Previous literature has shown that irrational traders can realize higher profits than rational ones in two ways: by obtaining market compensation for bearing higher risk than their rational counterparts; or by exploiting private information about fundamentals more aggressively and thereby intimidating rational traders. In contrast to these arguments, the irrational investors in our model earn positive expected profits without any private information that is inherently related to fundamentals, in a setting where risk-neutral market makers ensure that there is no market compensation for bearing risk. Further, these expected profits are inadvertently earned, in that they obtain in a setting where the irrational investors are price takers who naïvely ignore the feedback effect.

We also find that irrational investors influence real investment choices even though prices follow a random walk. In our model, resources flow into sectors that are viewed favorably by irrational investors and flow out of sectors that are viewed unfavorably. This resource misallocation reduces ex ante firm values even though prices follow a random walk. This means that standard tests of informational efficiency are unlikely to tell us much about whether or not financial market prices provide the appropriate signals for efficient allocation of resources within an economy.[18]

[18] It may be possible to detect the effects of irrational trading indirectly by examining the cross-sectional implications of our model. In particular, our model suggests that irrational investors will

More broadly, our approach suggests that irrationality, when combined with feedback from the stock market to real investment, can generate phenomena akin to what Keynes referred to as 'animal spirits' in the stock market. Although our analysis is focused on the firm level, we expect that there can exist significant feedback from stock prices to aggregate economic activity, which may potentially offer profit opportunities for irrational market timers. Specifically, it may be worthwhile to consider whether our framework can be extended to a setting where irrational investors who engage in market timing realize positive expected profits, and more importantly, create price and investment fluctuations that can induce business cycles.

be more active in sectors such as high tech, where feedback is likely to be strong because of the interdependence of firms within this sector. Weston (2001) estimates a microstructure model and finds evidence that noise traders are especially active in the technology-heavy Nasdaq market.

## Appendix

**Derivation of Equation (11):** The wealth of the early-informed trader, denoted by $W^E$, is

$$W^E = x_2 F - (x_2 - x_1)P_2 - x_1 P_1.$$

Let $\mu \equiv \theta + kE(\theta|P_1, P_2)$. Substituting for $x_2$ from (10) into the expression for $W^E$, we have

$$\begin{aligned} W^E &= \frac{(\mu - P_2)}{R\sigma_\epsilon^2}(\mu + \epsilon) - \frac{(\mu - P_2)}{R\sigma_\epsilon^2} P_2 - x_1(P_1 - P_2) + B_0 \\ &= \frac{(\mu - P_2)^2}{R\sigma_\epsilon^2} + \frac{(\mu - P_2)\epsilon}{R\sigma_\epsilon^2} - x_1(P_1 - P_2) + B_0. \end{aligned} \tag{26}$$

Now from the formula for the characteristic function of a normal distribution, if $u \sim N(\mu, \sigma^2)$, then $E(exp(vu)) = exp(\mu v + (1/2)\sigma^2 v^2)$. In our case, setting $u = W^E$, $v = -R$, and using the fact that, from the perspective of the early informed, the only unknown at date 2 is the random variable $\epsilon$, we have

$$E(-exp(-RW^E)|\phi_2) = -exp\{-R[B_0 - x_1 P_1 + x_1 P_2 + (\mu - P_2)^2/(2R\sigma_\epsilon^2)]\}. \tag{27}$$

It follows that at date 1, the early-informed traders maximize the derived expected utility of their date 2 wealth

$$E[[-exp\{-R[B_0 - x_1 P_1 + x_1 P_2 + (\mu - P_2)^2/(2R\sigma_\epsilon^2)]\}]|\phi_1]. \tag{28}$$

Now, (28) can be written as

$$-(2\pi\sigma_{P_2}^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left\{-R\left[B_0 - x_1 P_1 + x_1 P_2 + \frac{(\mu - P_2)^2}{(2R\sigma_\epsilon^2)}\right]\right. \\ \left. -\frac{1}{2}\frac{(P_2 - \bar{P}_2)^2}{\sigma_{P_2}^2}\right\} d(P_2 - \bar{P}_2). \tag{29}$$

Completing squares, the expression within the exponential above can be written as

$$-\left[\frac{1}{2}w^2 s + hw + l\right], \tag{30}$$

where

$$\begin{aligned} w &= P_2 - \bar{P}_2 \\ h &= Rx_1 - \tfrac{(\mu-\bar{P}_2)}{\sigma_\epsilon^2} \\ s &= \tfrac{1}{\sigma_{P_2}^2} + \tfrac{1}{\sigma_\epsilon^2} \\ l &= Rx_1(\bar{P}_2 - P_1) + \tfrac{(\mu-\bar{P}_2)^2}{2\sigma_\epsilon^2} + RB_0. \end{aligned}$$

Define $u \equiv \sqrt{s}w + h/\sqrt{s}$. Then, expression (30) becomes $-(1/2)u^2 + (1/2)h^2/s - l$. The Jacobian of the transformation from $w$ to $u$ is $s^{-\frac{1}{2}}$, and thus the integral (29) becomes

$$-[2\pi\sigma_{P_2}^2 s]^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}u^2 + \frac{1}{2}\frac{h^2}{s} - l\right) du$$

$$= -\frac{1}{(\sigma_{P_2}^2 s)^{\frac{1}{2}}} exp\left(\frac{1}{2}\frac{h^2}{s} - l\right). \tag{31}$$

Solving for the optimal $x_1$ by maximizing the above objective, we obtain (11). An identical technique allows us to derive (13). □

**Derivation of the Equilibrium Price Coefficients in(9) and (8):** Plugging for $E_r(P_2)$ and $E_n(P_2)$ into the expressions for $\tau_1$ and $\tau_2$, we have

$$\begin{aligned} \tau_1 = \theta &\left[\frac{Mr_1}{R}\left\{\frac{1}{v_r(P_2)} + \frac{k}{(1+k)v_\epsilon}\right\} + \frac{M}{Rv_\epsilon} + \frac{Nn_1}{Rv_n(P_2)}\right] \\ +\eta &\left[\frac{Mr_2}{R}\left\{\frac{1}{v_r(P_2)} + \frac{k}{(1+k)v_\epsilon}\right\} + \frac{Nn_2}{Rv_n(P_2)} + \frac{N}{Rv_\epsilon}\right] \\ +z_1 &\left[\frac{Mr_3}{R}\left\{\frac{1}{v_r(P_2)} + \frac{k}{(1+k)v_\epsilon}\right\} + \frac{Nn_3}{Rv_n(P_2)} + 1\right], \end{aligned}$$

which can be written as

$$\tau_1 = k_1\theta + k_2\eta + k_3 z_1.$$

We then have $E(\theta|\tau_1,\tau_2) = m_1\tau_1 + m_2\tau_2$. Since $P_2 = (1+k)E(\theta|\tau_1,\tau_2)$, by equating coefficients, we have

$$\begin{aligned} b_1 &= (1+k)m_1k_1 + (1+k)m_2 \\ b_2 &= (1+k)m_1k_2 + (1+k)m_2 \\ b_3 &= (1+k)m_1k_3 + (1+k)Rv_\epsilon m_2 \\ b_4 &= (1+k)m_2Rv_\epsilon. \end{aligned}$$

We solve for $P_1$ as

$$P_1 = E(\theta|\tau_1) + kE[E(\theta|\tau_1,\tau_2)|\tau_1] = E(\theta|\tau_1) + \left(\frac{k}{1+k}\right)E(P_2|\tau_1). \tag{32}$$

Define $D_1 = k_1^2v_\theta + k_2^2v_\eta + k_3^2v_z$ and $D_2 = k_1b_1v_\theta + k_2b_2v_\eta + k_3b_2v_z$. Then it follows from a simple application of the projection theorem that

$$\begin{aligned} a_1 &= \frac{k_1^2v_\theta + kk_1D_2/(1+k)}{D_1} \\ a_2 &= \frac{k_1k_2v_\eta + kk_2D_2/(1+k)}{D_1} \\ a_3 &= \frac{k_1k_3v_z + kk_3D_2/(1+k)}{D_1}. \end{aligned}$$

This completes the solution procedures for the price coefficients $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$. $\square$

**Proof of Proposition 1:** Part 1 follows from the fact that the sequence of prices $P_1, P_2$ and $F$ form a martingale, increments to which are serially uncorrelated. Part 2 is demonstrated by direct calculation, as illustrated in Figures 1 and 2. $\square$

**Proof of Proposition 2:** The covariance between the trades of the early-irrational investors and the terminal value is

$$\text{cov}[(\eta - P_1)/(Rv_\epsilon), \theta + kE(\theta|P_1,P_2)] = \frac{kb_2'v_\eta}{Rv_\epsilon}.$$

Since, from (15), $b_2'$ is positive if and only if $M < 0.5$, the first part of the proposition follows.

To show Part 2, note that the covariance between the trades of the late irrational investors and the terminal value is

$$\mathrm{cov}[(\eta - P_2)/(Rv_\epsilon), \theta + kE(\theta | P_1, P_2)].$$

This covariance can be expressed in terms of the price coefficients as

$$-b_1'(1+k)(1+ka_1')v_\theta + b_2'[1-(1+k)b_2']v_\eta - (1+k)(b_3'^2 + b_4'^2)v_z.$$

Substituting for the coefficients $b_1'$-$b_4'$ from (14)-(17), the covariance becomes

$$-v_\theta D^{-1}[k^2 v_\theta(M^2 v_\eta + R^2 v_\epsilon^2 v_z) + k\{2M^2 v_\eta v_\theta + 2MR^2 v_\epsilon^2 v_\eta v_z - R^2 v_\epsilon^2 v_z(v_\eta - 2v_\theta)\} + v_\theta(M^2 v_\eta + R^2 v_\epsilon^2 v_z].$$

As $M \to 0$, the expression in square brackets approaches

$$\frac{-v_\theta[k^2 v_\theta + k(2v_\theta - v_\eta) + v_\theta]}{R^2 v_\epsilon^2 v_z + v_\theta + v_\eta},$$

which is positive so long as $v_\eta$ is sufficiently large. $\square$

**Proof of Proposition 3:** The covariance in the first part is

$$\mathrm{cov}[(\eta - P_1)/(Rv_\epsilon), P_2 - P_1] = (1+k)b_2' v_\eta/(Rv_\epsilon),$$

which is positive so long as $b_2' > 0$. By (15), this is the case if and only if $M < 0.5$.

Similarly, the covariance in Part 2 is

$$\mathrm{cov}[(\eta - P_2)/(Rv_\epsilon), P_2 - P_1],$$

which is positive if and only if

$$b_2' v_\eta > (1+k)(a_1'^2 v_\theta + b_2'^2 v_\eta + (b_3'^2 + b_4'^2)v_z.$$

By (14)-(17), the above condition is true if and only if (19) holds.

Finally, the covariance in Part 3 is

$$\mathrm{cov}[(\eta - P_2)/(Rv_\epsilon), (\eta - P_1)/(Rv_\epsilon)],$$

which is positive if and only if

$$1 > 1 - b_2'(1+k),$$

so that (20) follows from (15). $\square$

**Proof of Proposition 4:** To prove part 1, we observe that prices form a martingale (as in the first part of Proposition 1) or perform an explicit calculation. In particular,

$$\mathrm{cov}(F - P_2, P_2 - P_1) = b_1'(1 - b_1')v_\theta - b_2'^2 v_\eta - (b_3'^2 + b_4'^2)v_z.$$

Substituting for the equilibrium values of $b_1'$, $b_2'$, $b_3'$, and $b_4'$, the first part follows.

To prove part 2, note that the date 1 order flow, denoted by $Q_1$, is $Q_1 = M\eta/(Rv_\epsilon) + z_1$, whereas the date 2 order flow, $Q_2$, is

$$Q_2 = (1 - M)(\eta - P_2)/(Rv_\epsilon) + (\theta - P_2)/(Rv_\epsilon) + z_2 - M\eta/(Rv_\epsilon).$$

Since $P_2 = (1 + k)(b_1'\theta + b_2'\eta + b_3'z_1 + b_4'z_2)$, we have

$$\mathrm{cov}(Q_1, Q_2) = Mv_\eta[\{1-M\}\{1-(1+k)b_2'\}-(1+k)b_2'-M]+Rv_\epsilon v_z[-(1+k)(1-M)b_3'-(1+k)b_3'].$$

Substituting for $b_2'$ and $b_3'$ from (15)-(16), the above covariance reduces to $Mv_\eta(1-2M)$, and part 2 follows.

Now let us consider part 3. The standard formula for a conditional covariance matrix of a normal vector $X_1 \sim N(0, \Sigma_1)$ conditional on another normal vector $X_2 \sim N(0, \Sigma_2)$

is

$$var(X_1|X_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21},$$

where $\Sigma_{ij}$ represents the covariance matrix between $X_i$ and $X_j$. Now, we have that $P_3 - P_2 = \theta - E(\theta|\tau_1, \tau_2)$ and $P_2 - P_1 = (1+k)E(\theta|\tau_1, \tau_2)$. Letting $X_1 = [P_3 - P_2, P_2 - P_1]$ and $X_2 = \eta$, we find that

$$\text{cov}(P_3 - P_2, P_2 - P_1|\eta) = \frac{R^4 v_\epsilon^4 v_\theta^2 v_z^2 (2M-1)^2(1+k)}{D^2},$$

which is always positive. Similarly, letting $X_1 = [P_3 - P_2, P_2 - P_1]$ and $X_2 = \theta$, we find that

$$\text{cov}(P_3 - P_2, P_2 - P_1|\theta) = -[1+k][(b_2'^2 v_\eta + (b_3'^2 + b_4'^2)v_z],$$

which is always negative. □

**Proof of Proposition 5:** This proposition follows directly from an examination of the expression on the right-hand side of (21). □

**Proof of Proposition 6:** Since $D$ is a common denominator in both (21) and (22), comparing the expected profits reduces to comparison of the numerators on the RHS of these equations. So the expected profits of the irrational traders exceed those of the rational traders if and only if

$$Rv_\epsilon v_\eta v_\theta v_z (1-2M)[(k+2)M - 1] > R^2 v_\epsilon^2 v_\theta v_z [v_\eta \{(2M-1)^2 + M^2\} + R^2 v_\epsilon^2 v_z].$$

Straightforward algebra shows the equivalence of the above condition to (23), so long as $M < 0.5$. □

**Proof of Proposition 7:** From (21) and (22), the expression for the profit differential, denoted by $\Delta\pi$, is

$$\Delta\pi = \frac{-Rv_\epsilon v_\theta v_z\left[kMv_\eta(2M-1) + M^2 v_\eta(5Rv_\epsilon + 4) - 4Mv_\eta(Rv_\epsilon + 1) + R^3 v_\epsilon^3 v_z + v_\eta(Rv_\epsilon + 1)\right]}{D}. \quad (33)$$

The derivative of the above expression with respect to $k$ is

$$\frac{MRv_\epsilon v_\eta v_\theta v_z(1-2M)}{D}.$$

This is positive if and only if $M < 0.5$.

Similarly, the derivative of the right-hand side of (33) with respect to $v_\eta$ is

$$\frac{R^3 v_\epsilon^3 v_z^2 v_\theta(1-2M)[kM(R^2 v_\epsilon^2 v_z + v_\theta) - (R^2 v_\epsilon v_z + Rv_\epsilon v_\theta + v_\theta)(1-2M)}{D^2}.$$

This is positive so long as $M < 0.5$ and (24) is satisfied.

Finally, differentiating the RHS of (33) with respect to $v_\theta$ yields

$$R^3 v_\epsilon^3 v_z^2[v_\eta\{(2M-1)^2+M^2\}+R^2 v_\epsilon^2 v_z][kMv_\eta(1-2M)-\{v_\eta(2M-1)^2(Rv_\epsilon+1)+M^2 v_\eta Rv_\epsilon\}-R^3 v_\epsilon^3 v_z,$$

which is positive so long as $M < 0.5$ and (25) holds. □

**Proof of Proposition 8:** Equation (2) indicates that the amount of capital allocated to the growth opportunity equals $E(\theta|P_1, P_2)$. The correlation between $\eta$ and the above quantity is positive if and only if $b_2'$ is positive, and this is true if and only if $M < 0.5$ (see equation (15)), proving Part 1 of the proposition.

It can be shown from (14)-(17) that the ex ante variance of the RHS of equation (2) is

$$D^{-1}\left[v_\theta^2(M^2 v_\eta + R^2 v_\epsilon^2 v_z\right].$$

The derivative of this quantity with respect to $v_\eta$ is

$$-D^{-2}\left[R^4 v_\epsilon^4 v_\theta^2 v_z^2(2M-1)^2\right],$$

which is always negative, proving Part 2. Part 3 follows from substituting the optimal value of $C$ from (2) into (1) and noting from (4) that the ex ante expectation of (1) is monotonically related to the ex ante variance of $C$. $\square$

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**Figure 1**
**Ex ante expected noise trader profits vs. feedback parameter**
**[var(theta)=.4,var(eta)=1.9,R=0.18,var(z)=.6,var(epsilon)=4.2,M=0.05,N=.12]**

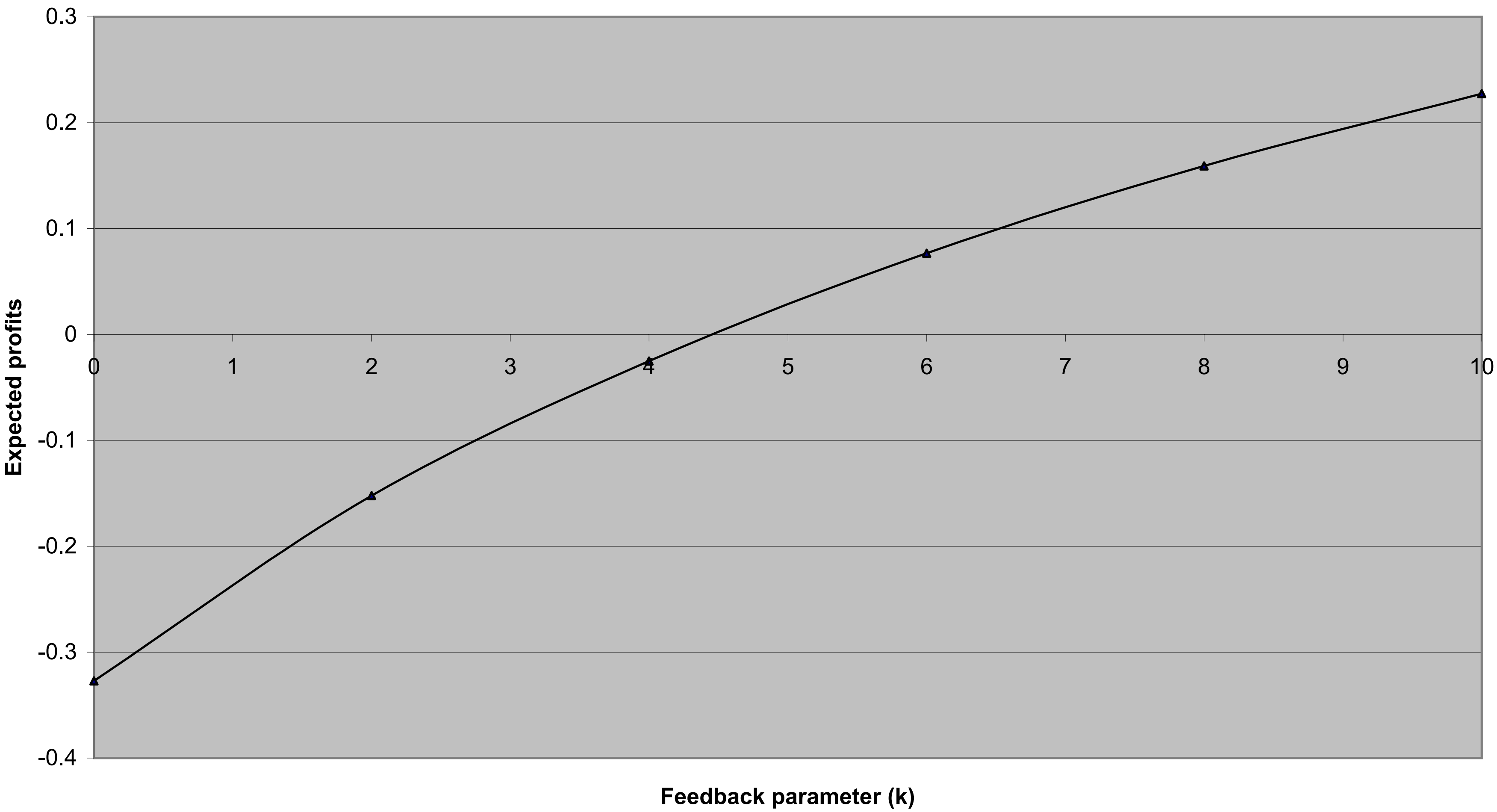

**Figure 2**
**Difference between expected profits of irrational and informed traders, as a proportion of the expected profits of informed traders**
**[var(theta)=0.1, var(eta)=3.14, R=0.21, var(z)=0.99, var(epsilon)=3.16, M=0.08, N=0.06]**