# Estimating a Model of Real and Nominal Term Structures using Treasury Yields, Inflation Forecasts, and Inflation Swap Rates

J. Haubrich *   G. Pennacchi†   P. Ritchken‡

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*Federal Reserve Bank of Cleveland. The views expressed here are those of the authors, and not necessarily those of the Federal Reserve Bank of Cleveland or the Federal Reserve System.

†College of Business, University of Illinois at Urbana-Champaign.

‡Weatherhead School of Management, Case Western Reserve University.

## Abstract

This paper develops and estimates an equilibrium model of the term structures of nominal and real interest rates. The factors include the short-term real interest rate, expected inflation, and inflation's central tendency. Additional factors are GARCH volatility processes that drive actual inflation, expected inflation, the real rate, and the central tendency. We derive analytical solutions for the prices of nominal bonds and inflation-indexed bonds that have an indexation lag, such as U.S. Treasury Inflation-Protected Securities (TIPS). Solutions for expected inflation rates and rates on inflation swaps also are derived. The model's parameters are estimated using data on nominal Treasury yields, survey forecasts of inflation, and inflation swap rates. We find that allowing for GARCH effects is particularly important for real interest rate and expected inflation processes, but that long-horizon real and inflation risk premia are relatively stable. Comparing our model prices of inflation indexed bonds to those of TIPS suggests that TIPS were underpriced prior to 2004 but subsequently were valued fairly.

# 1 Introduction

The Treasury yield curve provides a wealth of information, but for many purposes it is more important to know its key components: real rates, expected inflation, and real and inflation risk premia. In this paper we develop a model of the nominal and real term structure of interest rates and present an estimation technique that allows us to identify these components. These elements of the term structure may be of interest primarily to policymakers and others engaged in macroeconomics, but we also expect that a clearer view of yield curve characteristics will permit more accurate pricing of inflation-linked securities, such as inflation-indexed bonds and inflation derivatives. Due to the recent rise in inflation volatility, it is likely that demand for inflation-linked securities will grow.

We are able to derive analytical solutions for the prices of inflation-indexed (real) bonds that include an indexation lag, a feature found in all inflation-linked bonds.[1] Similarly, our model can price inflation swaps, which we use along with Treasury yields and survey forecasts of inflation to estimate the model's parameters. Zero coupon inflation swaps are the most liquid of inflation derivatives traded in the over-the-counter (OTC) market. Employing data on inflation swaps allows us to empirically distinguish between real and inflation risk premia.

The model we propose has other distinguishing features. It is a discrete time, multi-factor model that incorporates processes describing the characteristics of real rates and inflation. The volatility of shocks to real rates, inflation, expected inflation, and inflation's central tendency vary stochastically over time, following GARCH processes.[2] These volatilities also affect the risk premia associated with the shocks to these processes. The conditional normality of GARCH allows closed-form solutions, but over longer periods allows the distribution of interest rates to show significant skewness and kurtosis.

By modeling a stochastic central tendency for inflation, as well as stochastic volatility for real and inflation factors, we can capture changing monetary and real economic environments. Several recent papers forego the GARCH approach and introduce stochastic volatility via regime-switching (Ang and Bekaert 2002);(Dai, Singleton, and Yang 2003). The GARCH approach allows for substantial changes in volatilities without forcing those changes to fit into discrete "regimes." Even where distinct regimes are evident, the inflation and interest rate behavior can differ markedly between regimes of the same type (Bordo and Haubrich 2004). Furthermore, some variables of crucial interest to us, such as inflationary expectations, often show smooth transitions between regimes (Haubrich and Ritter 2000);(Haubrich and Ritter 2004).

[1] To the best of our knowledge, our paper is the first to explicitly account for the indexation lag when valuing inflation-indexed bonds.

[2] Term structure models incorporating state variables with GARCH volatility include (Cvsa and Ritchken 2001) and (Heston and Nandi 2003).

We obtain several noteworthy empirical results. First, we find that real interest rates are the most volatile component of the yield curve, and it is especially important to allow their volatility to display GARCH behavior. Real rates were negative for much of the 2002 to 2005 period, which may have helped inflate a credit bubble. Second, we find that expected inflation is negatively correlated with real rates, and it also shows statistically significant changes in volatility. Both real rates and expected inflation display rather strong mean reversion. Third, over our sample period of 1982 to 2008, inflation's central tendency, which can be viewed as investors' expectation of medium-horizon inflation, declined substantially. This is consistent with an increase in credibility regarding the Federal Reserve's desire to maintain low inflation.

Fourth, we find a real interest rate risk premium that is substantial and fairly stable, varying between 150 and 170 basis points for a ten-year maturity bond. The inflation risk premium on a ten-year bond varied between 38 and 60 basis points during our sample period. Lastly, by comparing our model's implied yields for inflation-indexed bonds to actual prices of U.S. Treasury Inflation-Protected Securities (TIPS), we document evidence that TIPS were underpriced prior to 2004, but subsequently appeared to be fairly valued.

The paper proceeds as follows. Section 2 introduces a model of real interest rates and inflation that is used to derive the term structures of nominal bonds, inflation forecasts, inflation-indexed bonds, and inflation swap rates. Section 3 describes the data used to calibrate and estimate and model's parameters, with Section 4 explaining the estimation technique. Section 5 describes the results and Section 6 concludes.

## 2 A Model of Nominal and Real Term Structures

Consider a discrete time environment with multiple periods, each of length $\Delta t$ measured in years. Let $\frac{M_{t+\Delta t}}{M_t}$ be the nominal pricing kernel with dynamics:

$$\frac{M_{t+\Delta t}}{M_t} = e^{-i_t \Delta t - \frac{1}{2}\sum_{j=1}^{4} \phi_j^2 h_{j,t}^2 \Delta t + \sum_{j=1}^{4} \phi_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t}} \tag{1}$$

Here $\epsilon_{j,t+\Delta t}, j = 1, 2, .., 4$ are independent standard normal random variables and $\phi_j h_{j,t}$, $j = 1, 2, .., 4$ are market prices of risk associated with these four sources of uncertainty. $h_{j,t}$, $j = 1, 2, .., 4$ represents four different volatility state variables whose dynamics will be specified later. Let $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$ denote the expectations operator conditional on information at date $t$, $\mathcal{F}_t$. If we value a one period payoff of \$1, then

$$E_t\left[\frac{M_{t+\Delta t}}{M_t}\right] = e^{-i_t \Delta t} \tag{2}$$

so that $i_t$ is the annualized, one period nominal interest rate.

Let the price index at date $t$ be $I_t$. For example, $I_t$ can denote the date $t$ Consumer Price Index (CPI). Its dynamics are assumed to satisfy:

$$\frac{I_{t+\Delta t}}{I_t} = e^{\pi_t \Delta t - \frac{1}{2} h_{1,t}^2 \Delta t + h_{1,t}\sqrt{\Delta t}\epsilon_{1,t+\Delta t}} \tag{3}$$

where the variable $\pi_t = \frac{1}{\Delta t}\ln\left(E_t\left[I_{t+\Delta t}/I_t\right]\right)$ is rate of expected inflation for the period $[t, t+\Delta t]$.

Given the processes for the nominal pricing kernel and the price index, we can compute the real pricing kernel, $m_t$. In particular:

$$\begin{aligned} \frac{m_{t+\Delta t}}{m_t} &= \frac{M_{t+\Delta t}}{M_t}\frac{I_{t+\Delta t}}{I_t} \\ &= e^{\left(\pi_t - i_t - \frac{1}{2}h_{1,t}^2\right)\Delta t - \frac{1}{2}\sum_{j=1}^{4}\phi_j^2 h_{j,t}^2 \Delta t + \sum_{j=1}^{4}\phi_j h_{j,t}\sqrt{\Delta t}\epsilon_{j,t+\Delta t} + h_{1,t}\sqrt{\Delta t}\epsilon_{1,t+\Delta t}} \end{aligned} \tag{4}$$

Taking expectations on the left-hand-side of (4) defines $r_t$, the one period real interest rate:

$$E_t\left[\frac{m_{t+\Delta t}}{m_t}\right] = e^{-r_t \Delta t} \tag{5}$$

Taking expectations on the right-hand-side of (4) and equating to (5), we obtain:

$$e^{-r_t\Delta t} = e^{\left(-i_t + \pi_t + h_{1,t}^2\phi_1\right)\Delta t}, \tag{6}$$

from which

$$i_t = \pi_t + r_t + \phi_1 h_{1,t}^2 \tag{7}$$

All that remains for pricing to proceed is to specify the dynamics of the state variables. It is assumed that

$$\begin{aligned} \pi_{t+\Delta t} - \pi_t &= \left[\alpha_t + a_1 r_t + a_2 \pi_t\right]\Delta t + \sqrt{\Delta t}\sum_{j=1}^{2}\beta_j h_{j,t}\epsilon_{j,t+\Delta t} \\ r_{t+\Delta t} - r_t &= \left[b_0 + b_1 r_t + b_2 \pi_t\right]\Delta t + \sqrt{\Delta t}\sum_{j=1}^{3}\gamma_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \alpha_{t+\Delta t} - \alpha_t &= \left[c_0 + c_1 \alpha_t\right]\Delta t + \sqrt{\Delta t}\sum_{j=1}^{4}\rho_j h_{j,t}\epsilon_{j,t+\Delta t} \\ h_{j,t+\Delta t}^2 - h_{j,t}^2 &= \left[d_{j0} + d_{j1}h_{j,t}^2 + d_{j2}\left(\epsilon_{j,t+\Delta t} - d_{j3}h_{j,t}\right)^2\right]\Delta t,\ j = 1, ..., 4 \end{aligned} \tag{8}$$

where $\alpha_t$ is an additional state variable that shifts the future path of the expected inflation rate. The first three equations in (8) satisfy a first-order vector autoregression. Subject to parameter stationarity conditions, the constants in these equations can be related to the unconditional mean (steady-state level) of expected inflation, $\overline{\pi}$, and the unconditional mean of the real rate, $\overline{r}$. These relationships are

$$\overline{\pi} = -\frac{a_1 b_0 c_1 + b_1 c_0}{(a_1 b_2 - a_2 b_1)\, c_1} \tag{9}$$

$$\overline{r} = \frac{a_2 b_0 c_1 + b_2 c_0}{(a_1 b_2 - a_2 b_1)\, c_1} \tag{10}$$

The unconditional mean of $\alpha_t$ is $-c_0/c_1 = -(a_1\overline{r} + a_2\overline{\pi})$. If a constant is added to $\alpha_t$ such that $\widehat{\alpha}_t \equiv \alpha_t + a_1\overline{r} + (1 + a_2)\,\overline{\pi}$, then the unconditional mean of $\widehat{\alpha}_t$ equals $\overline{\pi}$, and $\widehat{\alpha}_t$ is commonly referred to as the "central tendency" of the expected inflation rate.[3] It equals the current mean reversion level or target level to which the inflation rate is expected to tend. For simplicity, we shall refer to $\alpha_t$ as the central tendency, but it should be understood that it differs from the true central tendency, $\widehat{\alpha}_t$, by a constant.

The equations in (3) and (8) specify that actual inflation, expected inflation, the real interest rate, and inflation's central tendency follow imperfectly correlated processes having stochastic volatilities. Without loss of generality, we can restrict $\beta_2 = \gamma_3 = \rho_4 = 1$.[4] From (3), the one-period inflation rate, $\ln[I_{t+\Delta t}/I_t]$, is assumed to have an annualized standard deviation, $h_{1,t}$, that follows a GARCH process driven by the inflation innovation, $\epsilon_t$. The first equation in (8) permits the expected inflation rate, $\pi_t$, to follow a mean reverting process that tends toward a target level or central tendency which is itself follows a mean reverting process. The expected inflation innovations depend on surprises to actual inflation, $h_{1,t}\epsilon_1$, and an orthogonal shock, $h_{2,t}\ \epsilon_2$. The volatility of the expected inflation process is stochastic, linked to the volatility of inflation innovations, and also to the specific expected inflation innovations whose standard deviation, $h_{2,t}$, follows a GARCH process driven by $\epsilon_{2,t}$.

The process for the real interest rate, $r_t$, mean reverts to $\overline{r}$ and has an orthogonal shock, $h_{3,t}\epsilon_3$. Like the other state variables, the volatility of the real rate innovation, $h_{3,t}$, follows a GARCH process that depends on $\epsilon_3$. Finally, the process for inflation's central tendency, $\alpha_t$, is correlated with actual inflation, expected inflation, and real rates, but also has its unique shock $h_{4,t}\epsilon_4$ that satisfies a GARCH process determined by $\epsilon_4$. Note that from the pricing kernel equation (1), each of the four shocks $h_{1,t}\epsilon_1$, $h_{2,t}\epsilon_2$, $h_{3,t}\epsilon_3$, and $h_{4,t}\epsilon_4$ commands a risk premium of $\phi_1 h_{1,t}$, $\phi_2 h_{2,t}$, $\phi_3 h_{3,t}$, and $\phi_4 h_{4,t}$, respectively.

The dynamics of the $h_{j,t}$, $j = 1, ..., 4$, follow the Nonlinear Asymmetric GARCH model of Engle and Ng (Engle and Ng 1993). Subject to stationarity conditions, the steady-state levels

[3] Term structure models specifying a central tendency include (Hull and White 1994), (Jegadeesh and Pennacchi 1996), and (Balduzzi, Das, and Foresi 1998).

[4] These three restrictions permit the identification of the levels of the stochastic volatilities $h_{2,t}$, $h_{3,t}$, and $h_{4,t}$.

of these processes are

$$\overline{h}_j^2 = -\frac{d_{j0} + d_{j2}}{d_{j1} + d_{j2}d_{j3}^2} \ , \quad j = 1, ..., 4 \tag{11}$$

The model extends a GARCH(1,1) to allow for asymmetric responses to the innovations when the parameters $d_{j3}$, $j = 1, ..., 4$ are non-zero. When $d_{j3}$ is positive (*negative*), negative values of $\epsilon_{j,t}$ have a larger (*smaller*) impact on $h^2_{j,t+\Delta t}$ than do positive values. Collectively, the $h_{j,t}$ act as scaling factors that determine the local volatilities for inflation, expected inflation, real rates, and the central tendency. Of course, if all these GARCH effects are shut down, then there will be no stochastic volatility, and the model will reduce to a Markovian model with three state variables.[5] With stochastic volatility, our model has four stochastic drivers and seven state variables. While the one-period distribution of the state variables $\pi_t$, $r_t$, and $\alpha_t$ are conditionally normal, over multiple periods, the distribution will not be normal. The parameters $d_{j2}$ and $d_{j3}$, $j = 1, ..., 4$ heavily influence the skewness and kurtosis in the distribution of yields to maturity over multiple periods.

Having specified the dynamics of the state variables affecting the nominal pricing kernel and inflation, we now turn to valuing securities that have nominal and real payoffs. We also derive inflation expectations.

## 2.1 The Term Structure of Nominal Interest Rates

Let $P(t, t + n\Delta t)$ be the date $t$ price of a nominal bond that pays \$1 at date $t + n\Delta t$, where $n$ is a non-negative integer. We have:

$$P(t, t + n\Delta t) = E_t \left[ \frac{M_{t+\Delta t}}{M_t} P(t + \Delta t, t + n\Delta t) \right] \tag{12}$$

Proposition 1 below provides the expressions for the term structure of nominal interest rates.

**Proposition 1** *Under the above dynamics, nominal bond prices are given by the following recursive equation:*

$$P(t, t + n\Delta t) = e^{-K_n - A_n\pi_t - B_n r_t - C_n\alpha_t - \sum_{j=1}^{4} D_{j,n}h^2_{j,t}} \qquad \textit{for } n \geq 1. \tag{13}$$

*where* $K_1 = 0$*,* $A_1 = \Delta t$*,* $B_1 = \Delta t$*,* $C_1 = 0$*,* $D_{1,1} = \phi_1 \Delta t$*,* $D_{j,1} = 0$ *for* $j = 2, 3, 4$*, and*

[5] The homoskedastic (constant volatility) case occurs when $d_{j,1} = -1/\Delta t$ and $d_{j,2} = d_{j,3} = 0$. This would correspond to a multivariate (Vasicek 1977) model as developed in (Langetieg 1980). (Pennacchi 1991) derives a special case of this model from a monetary production economy where expected inflation, $\pi_t$, and the real interest rate, $r_t$, are the only two state variables.

$$\begin{aligned}
K_{n+1} &= K_n + \left(b_0 B_n + c_0 C_n + \textstyle\sum_{j=1}^4 d_{j0} D_{j,n}\right)\Delta t + \frac{1}{2}\textstyle\sum_{j=1}^4 \ln(1 + 2d_{j2}\Delta t D_{j,n}) \\
A_{n+1} &= \Delta t + (1 + a_2 \Delta t)\, A_n + b_2 \Delta t B_n \\
B_{n+1} &= \Delta t + a_1 \Delta t A_n + (1 + b_1 \Delta t)\, B_n \\
C_{n+1} &= \Delta t A_n + (1 + c_1 \Delta t)\, C_n \\
D_{j,n+1} &= \left[1 + \left(d_{j1} + d_{j2} d_{j3}^2\right)\Delta t\right] D_{j,n} + \frac{1}{2}\phi_j^2 \Delta t - Q_j \Delta t
\end{aligned}$$

and

$$\begin{aligned}
Q_1 &= \frac{(\phi_1 - A_n\beta_1 - B_n\gamma_1 - C_n\rho_1 + 2D_{1,n} d_{12} d_{13}\sqrt{\Delta t})^2}{2(1 + 2D_{1,n} d_{12}\Delta t)} - \phi_1 \\
Q_2 &= \frac{(\phi_2 - A_n\beta_2 - B_n\gamma_2 - C_n\rho_2 + 2D_{2,n} d_{22} d_{23}\sqrt{\Delta t})^2}{2(1 + 2D_{2,n} d_{22}\Delta t)} \\
Q_3 &= \frac{(\phi_3 - B_n\gamma_3 - C_n\rho_3 + 2D_{3,n} d_{32} d_{33}\sqrt{\Delta t})^2}{2(1 + 2D_{3,n} d_{32}\Delta t)} \\
Q_4 &= \frac{(\phi_4 - C_n\rho_4 + 2D_{4,n} d_{42} d_{43}\sqrt{\Delta t})^2}{2(1 + 2D_{4,n} d_{42}\Delta t)}
\end{aligned}$$

*Proof*: See the Appendix.

## 2.2 Expectations of Inflation

Let $I(t, t+n\Delta t)$ be the date $t$ forecast of growth in the price level over the time period $[t, t+n\Delta t]$. Then:

$$I(t, t + n\Delta t) = E_t\left[\frac{I_{t+n\Delta t}}{I_t}\right] \tag{14}$$

**Proposition 2**

*The date $t$ expectation of inflation for a horizon of $n$ periods is*

$$I(t, t + n\Delta t) = e^{\bar{K}_n + \bar{A}_n \pi_t + \bar{B}_n r_t + \bar{C}_n \alpha_t + \sum_{j=1}^4 \bar{D}_{j,n} h_{j,t}^2} \quad \text{for } n \geq 1. \tag{15}$$

*where* $\bar{K}_1 = 0,\ \bar{A}_1 = \Delta t,\ \bar{B}_1 = 0,\ \bar{C}_1 = 0,\ \bar{D}_{j,1} = 0$ *for* $j = 1, 2, 3, 4,$ *and*

$$\begin{aligned}
\bar{K}_{n+1} &= \bar{K}_n + \left(b_0\bar{B}_n + c_0\bar{C}_n + \textstyle\sum_{j=1}^4 d_{j0}\bar{D}_{j,n}\right)\Delta t - \frac{1}{2}\textstyle\sum_{j=1}^4 \ln(1 - 2d_{j2}\Delta t\bar{D}_{j,n}) \\
\bar{A}_{n+1} &= \Delta t + (1 + a_2\Delta t)\bar{A}_n + b_2\Delta t\bar{B}_n \\
\bar{B}_{n+1} &= a_1\Delta t\bar{A}_n + (1 + b_1\Delta t)\bar{B}_n \\
\bar{C}_{n+1} &= \Delta t\bar{A}_n + (1 + c_1\Delta t)\bar{C}_n \\
\bar{D}_{j,n+1} &= \bar{D}_{j,n}\left[1 + \left(d_{j1} + d_{j2}d_{j3}^2\right)\Delta t\right] + \bar{Q}_j\Delta t
\end{aligned}$$

and

$$\begin{aligned}
\bar{Q}_1 &= \frac{(1 + \bar{A}_n\beta_1 + \bar{B}_n\gamma_1 + \bar{C}_n\rho_1 - 2\bar{D}_{1,n}d_{12}d_{13}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{1,n}d_{12}\Delta t)} - \frac{1}{2} \\
\bar{Q}_2 &= \frac{(\bar{A}_n\beta_2 + \bar{B}_n\gamma_2 + \bar{C}_n\rho_2 - 2\bar{D}_{2,n}d_{22}d_{23}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{2,n}d_{22}\Delta t)} \\
\bar{Q}_3 &= \frac{(\bar{B}_n\gamma_3 + \bar{C}_n\rho_3 - 2\bar{D}_{3,n}d_{32}d_{33}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{3,n}d_{32}\Delta t)} \\
\bar{Q}_4 &= \frac{(\bar{C}_n\rho_4 - 2\bar{D}_{4,n}d_{42}d_{43}\sqrt{\Delta t})^2}{2(1 - 2\bar{D}_{4,n}d_{42}\Delta t)}
\end{aligned}$$

*Proof*: See the Appendix

Proposition 2 provides the expectation of inflation starting from the current date $t$. Because our data also contains survey forecasts of an inflation rate that begins and ends at two future dates, it is useful to derive an expression for such a forecast. Let $t$ be the current date, $t + n_1\Delta t$ be the date at which the inflation forecast starts, and $t + n_2\Delta t$ be the date at which the inflation forecast ends, where $n_2 > n_1$. Let $m \equiv n_2 - n_1$, for example, $m = 3$ periods (months) would occur if the forecast is of an inflation rate over a future quarter of a year. If survey participants forecast a continuously-compounded rate, then their date $t$ forecast is

$$E_t\left[\frac{1}{m\Delta t}\ln\left(\frac{I_{t+n_2\Delta t}}{I_{t+n_1\Delta t}}\right)\right] = \frac{1}{m\Delta t}\left(E_t\left[\ln\left(\frac{I_{t+n_2\Delta t}}{I_t}\right)\right] - E_t\left[\ln\left(\frac{I_{t+n_1\Delta t}}{I_t}\right)\right]\right) \tag{16}$$

**Proposition 3**

$$E_t\left[\ln\left(\frac{I_{t+n\Delta t}}{I_t}\right)\right] = K_n^* + A_n^*\pi_t + B_n^*r_t + C_n^*\alpha_t + \textstyle\sum_{j=1}^4 D_{j,n}^*h_{j,t}^2 \tag{17}$$

*where*

$$
\begin{aligned}
K^*_{n+1} &= K^*_n + \left(b_0 B^*_n + c_0 C^*_n + \textstyle\sum_{j=1}^4 (d_{j0} + d_{j2}) D^*_{j,n}\right) \Delta t \\
A^*_{n+1} &= \Delta t + (1 + a_2 \Delta t) A^*_n + b_2 \Delta t B^*_n \\
B^*_{n+1} &= a_1 \Delta t A^*_n + (1 + b_1 \Delta t) B^*_n \\
C^*_{n+1} &= \Delta t A^*_n + (1 + c_1 \Delta t) C^*_n \\
D^*_{j,n+1} &= D^*_{j,n} \left[1 + (d_{j1} + d_{j2} d^2_{j3}) \Delta t\right] - 1_{j=1} \frac{1}{2} \Delta t
\end{aligned}
$$

*where* $1_{j=1} = 1$ *if* $j = 1$ *and* $0$ *otherwise, and* $K^*_1 = 0$, $A^*_1 = \Delta t$, $B^*_1 = 0$, $C^*_1 = 0$, $D^*_{1,1} = -\frac{1}{2}\Delta t$, *and* $D^*_{j,1} = 0$, *for* $j = 2, 3, 4$.

### 2.3 Pricing of Treasury Inflation Securities

Treasury Inflation-Protected Securities (TIPS) have payoffs linked to an inflation index, thereby protecting investors from the adverse effects of inflation. Since a coupon-bearing TIPS can be decomposed into a portfolio of zero-coupon TIPS contracts, it is sufficient to value a zero-coupon TIPS contract. In practice, a TIPS contract does not provide full coverage against inflation. Rather, the inflation index for a TIPS payment is based on the Consumer Price Index (CPI) recorded at a date prior to the bond's date of payment. One reason for this is that the CPI is not revealed immediately at the date for which it is recorded, but is reported with a lag. In the US, for example, a payout for TIPS is based on the CPI index recorded at a date three months ($\frac{1}{4}$ year) prior to the bond's payment date.

Since this indexation lag feature can be important, we define $V^d(t; t_s, t_e)$ to be the date $t$ value of a zero-coupon TIPS contract that pays an amount linked to the price index recorded at date $t_e$ which is $n$ periods of length $\Delta t$ in the future. Thus, the bond payoff is based on the price index at date $t_e = t + n\Delta t$, i.e. $I_{t+n\Delta t}$, but the actual payment date, $t_p$ is $d$ periods later at date $t_p = t_e + d\Delta t$. Therefore, $d$ is the indexation lag. Following actual practice, the price index at the initiation date is also lagged by $d$ periods. Let $t_s$ represent the date at which the initial index is recorded. Then, if $t$ is the initiation date, $t_s = t - d\Delta t$ and the TIPS payment at date $t_p$ equals $I_{t_e}/I_{t_s}$. Now, note that at date $t_e = t + n\Delta t$ we can value the payment to be made $d$ periods later as:

$$
V^d(t_e; t_s, t_e) = \frac{I_{t_e}}{I_{t_s}} P(t_e, t_e + d\Delta t). \tag{18}
$$

and at date $t$ we have:

$$
V^d(t; t_s, t_e) = E_t\left[\frac{M_{t+\Delta t}}{M_t} V^d(t + \Delta t; t_s, t_e)\right] \tag{19}
$$

The following Proposition provides the recursive equation for pricing TIPS with an indexation lag of $d$ periods.

**Proposition 4**

*The date $t$ value of a zero coupon TIPS which is indexed off the start date of $t_s$, has a payout determined by the index at the end date $t_e$, and pays out, with a delay of $d\Delta t$ years, at date $t_e + d\Delta t$, is given by*

$$V^d(t = t_e - n\Delta t; t_s, t_e) = \frac{I_t}{I_{t_s}} e^{-\tilde{K}_n - \tilde{A}_n \pi_t - \tilde{B}_n r_t - \tilde{C}_n \alpha_t - \sum_{j=1}^4 \tilde{D}_n h_{j,t}^2} \quad \text{for } n \geq 0, \text{ and } t \geq t_s + d\Delta t \tag{20}$$

*where* $\tilde{K}_0 = K_d$, $\tilde{A}_0 = A_d$, $\tilde{B}_0 = B_d$, $\tilde{C}_0 = C_d$, $\tilde{D}_{j,0} = D_{j,d}$ *for* $j = 1, 2, 3, 4$, *and*

$$\begin{aligned}
\tilde{K}_{n+1} &= \tilde{K}_n + (b_0 \tilde{B}_n + c_0 \tilde{C}_n + \textstyle\sum_{j=1}^4 d_{j,0} \tilde{D}_{j,n})\Delta t + \frac{1}{2} \textstyle\sum_{j=1}^4 \ln(1 + 2 d_{j2} \tilde{D}_{j,n} \Delta t) \\
\tilde{A}_{n+1} &= (1 + a_2 \Delta t)\tilde{A}_n + b_2 \Delta t \tilde{B}_n \\
\tilde{B}_{n+1} &= (1 + a_1 \tilde{A}_n)\Delta t + (1 + b_1 \Delta t)\tilde{B}_n \\
\tilde{C}_{n+1} &= \Delta t \tilde{A}_n + (1 + c_1 \Delta t)\tilde{C}_n \\
\tilde{D}_{j,n+1} &= \frac{1}{2}\phi_j^2 \Delta t + \tilde{D}_{j,n}(1 + d_{j1}\Delta t + d_{j2} d_{j3}^2 \Delta t) - \tilde{Q}_j \Delta t
\end{aligned}$$

*and*

$$\begin{aligned}
\tilde{Q}_1 &= \frac{(1 + \phi_1 - \tilde{A}_n \beta_1 - \tilde{B}_n \gamma_1 - \tilde{C}_n \rho_1 + 2\tilde{D}_{1,n} d_{12} d_{13} \sqrt{\Delta t})^2}{2(1 + 2\tilde{D}_{1,n} d_{12} \Delta t)} - (\phi_1 + \frac{1}{2}) \\
\tilde{Q}_2 &= \frac{(\phi_2 - \tilde{A}_n \beta_2 - \tilde{B}_n \gamma_2 - \tilde{C}_n \rho_2 + 2\tilde{D}_{2,n} d_{22} d_{23} \sqrt{\Delta t})^2}{2(1 + 2\tilde{D}_{2,n} d_{22} \Delta t)} \\
\tilde{Q}_3 &= \frac{(\phi_3 - \tilde{B}_n \gamma_3 - \tilde{C}_n \rho_3 + 2\tilde{D}_{3,n} d_{32} d_{33} \sqrt{\Delta t})^2}{2(1 + 2\tilde{D}_{3,n} d_{32} \Delta t)} \\
\tilde{Q}_4 &= \frac{(\phi_4 - \tilde{C}_n \rho_4 + 2\tilde{D}_{4,n} d_{42} d_{43} \sqrt{\Delta t})^2}{2(1 + 2\tilde{D}_{4,n} d_{42} \Delta t)}
\end{aligned}$$

*Proof:* See the Appendix

## 2.4 Zero Coupon Inflation Swaps

Zero coupon inflation swaps are the most liquid of all inflation derivative contracts that trade in the over-the-counter (OTC) market. They are quoted with maturities ranging from 1 to 30 years. In addition, inflation swaps serve as the basic building blocks for the pricing of the majority of other inflation-related derivatives.

A zero coupon inflation swap is a contract whereby the inflation buyer pays a predetermined fixed nominal rate and in return receives from the seller an inflation linked payment. At the initiation date, $t_0$, the (consumer) price index is initialized to its value at the date $d\Delta t = 0.25$ years earlier, say $t_s = t_0 - d\Delta t$. The ending date for the price index is denoted $t_e$, and the cash settlement or payment date is $t_p$ where $t_p = t_e + d\Delta t$. At this final date a fixed payment is exchanged for $I_{t_e}/I_{t_s}$, which is the inflation over the period $[t_s, t_e]$. The fixed payment is denoted $(1+k)^{t_e - t_s}$ where $k$ is the annually-compounded inflation swap rate. Thus, the net fixed for inflation swap payment is $(1+k)^{t_e-t_s} - I_{t_e}/I_{t_s}$.

Viewed from date $t_0$ the value of the fixed (nominal) leg is simply

$$V_{fix}(t_0) = P(t_0, t_p)(1+k)^{t_e - t_s}. \tag{21}$$

The payout of the inflation leg, $V_{inf}(t_0)$ say, equals the payout of a zero coupon TIPS, with payouts at date $t_p$ linked to the index values at dates $t_s$ and $t_e$:

$$V_{inf}(t_0) = V^d(t_0; t_s, t_e) \tag{22}$$

At the initiation date, $t_0$, the fair inflation swap rate is the value $k$ that equates $V_{fix}(t_0)$ with $V_{inf}(t_0)$. The resulting value $k^*(t_0; t_s, t_e)$ is given by:

$$k^*(t_0; t_s, t_e) = \left( \frac{V^d(t_0; t_s, t_e)}{P(t_0, t_p)} \right)^{1/(t_e - t_s)} - 1. \tag{23}$$

# 3 Data Description

Estimation of our model uses monthly data on U.S. Treasury security yields, survey forecasts of inflation, rates of actual (realized) inflation, and inflation swap rates. From (12), Treasury prices (and, therefore, yields) reflect all of the model's parameters, but they, alone, cannot identify the parameters of the real interest rate process from the actual and expected inflation processes. Therefore, data on actual inflation and survey forecasts of inflation are critical for distinguishing between real and nominal processes. However, note from (3) and (17) that these inflation-related data on the physical process for inflation do not reflect the market price of risk parameters, $\phi_j$, $j = 1, 2, .., 4$. Thus, it is important to augment our data with information that can help identify the different sources of risk premia. As can be seen from (23) and (20), inflation swap rates serve this purpose.

Most data series are available over the period January 1982 to June 2008, though the data on inflation swap rates starts in only April 2003. Treasury security yields are obtained from three sources. First, we obtain zero coupon Treasury yields of 1, 2, 3, 5, 7, 10, and 15 years to maturity from daily off-the-run Treasury yield curves recently constructed by Refet Gurkaynak, Brian Sack and Jonathan Wright (Gurkaynak, Sack, and Wright 2007).[6] Second, we collect daily secondary market yields for 3-month and 6-month Treasury bills from the Federal Reserve System's H.15 Release.[7] Third, we obtain a 1 month (30-day) Treasury bill yield from the Center for Research in Security Prices (CRSP).[8] All of the Treasury yields are taken as of the first trading day of each month.

Survey forecasts of Consumer Price Index (CPI) inflation come from two different sources. First, a monthly series beginning in 1982 is obtained from Blue Chip Economic Indicators.[9] At the beginning of each month, participants in this survey forecast future CPI inflation for quarterly time periods, starting from the current quarter going out to at most 8 quarters (2 years) in the future. For January, February, and March, inflation rate forecasts for 8 future quarters are made. For April, May, and June, forecasts for 7 future quarters are made. For July, August, and September, forecasts for 6 future quarters are made, while for October, November, and December, forecasts for 5 future quarters are made. We use Blue Chip's reported "consensus" forecast which is the average of the participants' forecasts. Second, we use the median forecast of CPI inflation over the next ten years made by participants of the Survey of Professional Forecasters (SPF), currently conducted by the Federal Reserve Bank of Philadelphia.[10] This 10-year forecast is at a quarterly frequency, and starts in December of 1991. Thus, we observe this forecast at the beginning of March, June, September, and December.[11]

Our estimation method also uses a quarterly time series of actual (realized) inflation rates. We constructed this monthly series of actual CPI inflation to correspond with the monthly CPI inflation forecasts.[12]

[6] Their daily Treasury yield curves are available from 1961 to the present and can be downloaded from http://www.federalreserve.gov/pubs/feds/2006. These zero-coupon yields were fitted by the method of Lars Svensson (Svensson 1994) using prices of off-the-run Treasury coupon notes and bonds.

[7] These data series can be obtained at http://research.stlouisfed.org/fred2/categories/116.

[8] CRSP provides a consistent time series for the one-month Treasury yield over the entire 1982 to 2008 period. A single time series was not available from Federal Reserve sources.

[9] The approximately 50 participants in this survey are economists employed by financial institutions, non-financial corporations, and research organizations.

[10] This survey was originally performed by the American Statitiscal Association and the National Bureau of Economic Research. The data is available at http://www.philadelphiafed.org/econ/spf/ .See (Croushore 1993) and (Stark 2004) for more details on the survey methods. SPF forecasts appear to be rational expectations of inflation in that they appear to incorporate all available information (Keane and Runkle 1990). A recent study (Ang, Bekaert, and Wei 2007a) finds that SPF forecasts significantly outperform a wide variety of other methods for predicting inflation.

[11] SPF participants make forecasts at approximately, the middle of February, May, August, and November of each year. To align this survey with our other data, we presume these forecasts come at the start of the next month.

[12] Since survey participants are asked to forecast the seasonally-adjusted CPI inflation rate, our monthly time series is also based on the seasonally-adjusted CPI. This data is available at http://research.stlouisfed.org/fred2/categories/9. However, it should be noted that TIPS and zero-coupon inflation swaps are indexed to the CPI that is not seasonally-adjusted. This difference is unlikely to have much impact on TIPS prices and swap rates, except perhaps for those with very short times to maturity. The variation in the

Finally, we obtained inflation swap rates for annual maturities from 2 to 10 years, as well as 12-, 15-, 20-, and 30-year maturities. All inflation swap rates are for the first trading day of each month. The 2- to 10-year swap maturities start in April of 2003, the 12-, 15-, and 20-year inflation swap rates start in November 2003, and the 30-year inflation swap rates start in March 2004.

## 4 Estimation Technique

The empirical technique imposes our model's restrictions on both the cross-sectional and time-series properties of bond yields, inflation, inflation forecasts, and inflation swap rates. Given that our data is observed at a monthly frequency, the model's period is taken to be $\Delta t = \frac{1}{12}$ of a year. This implies that the model's nominal short rate, $i_t$, is the one-month Treasury bill rate. Similarly, $\pi_t$ is the rate of expected inflation over the next month, and $r_t$ is the one-month real interest rate. Note that while these rates correspond to a one-month horizon, we express them in annualized terms.

Our method is similar to that used for maximum likelihood estimation of GARCH models, except that we allow for measurement error in most of the Treasury yields, inflation rate forecasts, and inflation swap rates. Denote $y_t(n_i)$ as the annualized, continuously-compounded yield observed at date $t$ on a nominal bond maturing in $n_i$ months. We assume

$$\begin{aligned} y_t\left(n_i\right) &= -\frac{1}{n_i \Delta t} \ln\left[P\left(t, t + n_i \Delta t\right)\right] + \omega_{t,i} \qquad (24) \\ &= \frac{1}{n_i \Delta t}\left[K_n + A_n \pi_t + B_n r_t + C_n \alpha_t + \textstyle\sum_{j=1}^4 D_{j,n} h_{j,t}^2\right] + \omega_{t,i} \end{aligned}$$

where $\omega_{t,i}$ is an independent measurement error distributed $N\left(0, w^2\right)$.

Similarly, let $s_t\left(n_{t,i}^b, n_{t,i}^e\right)$ be the annualized, continuously-compounded expected inflation rate over the period from the beginning date $t + n_{t,i}^b \Delta t$ to the ending date $t + n_{t,i}^e \Delta t$ reported from the survey at date $t$ by Blue Chip Economic Indicators.[13] It is assumed to take the form

$$\begin{aligned} s_t\left(n_{t,i}^b, n_{t,i}^e\right) &= \frac{1}{\left(n_{t,i}^e - n_{t,i}^b\right)\Delta t}\left(E_t\left[\ln\left(\frac{I_{t+n_{t,i}^e \Delta t}}{I_t}\right)\right] - E_t\left[\ln\left(\frac{I_{t+n_{t,i}^b \Delta t}}{I_t}\right)\right]\right) + \upsilon_{t,i} \qquad (25) \\ &= \frac{1}{\Delta n_{t,i} \Delta t}\left[\Delta K_{t,i}^* + \Delta A_{t,i}^* \pi_t + \Delta B_{t,i}^* r_t + \Delta C_{t,i}^* \alpha_t + \textstyle\sum_{j=1}^4 \Delta D_{t,j,i}^* h_{j,t}^2\right] + \upsilon_{t,i} \end{aligned}$$

CPI due to seasonal adjustments is likely to be small compared to other sources of CPI variation, particularly for medium and longer-term horizons.

[13] We convert the annualized, quarterly-compounded rates reported in the survey to annualized, continuously-compounded rates.

where $\upsilon_{t,i}$ is an independent measurement error distributed $N\left(0, \upsilon^2\right)$, $\Delta n_{t,i} = n_{t,i}^e - n_{t,i}^b$, $\Delta K_{t,i}^* \equiv K_{n_{t,i}^e}^* - K_{n_{t,i}^b}^*$, $\Delta A_{t,i}^* \equiv A_{n_{t,i}^e}^* - A_{n_{t,i}^b}^*$, $\Delta B_{t,i}^* \equiv B_{n_{t,i}^e}^* - B_{n_{t,i}^b}^*$, $\Delta C_{t,i}^* \equiv C_{n_{t,i}^e}^* - C_{n_{t,i}^b}^*$, and $\Delta D_{t,j,i}^* \equiv D_{j,n_{t,i}^e}^* - D_{j,n_{t,i}^b}^*$. For the monthly forecasts taken from Blue Chip Economic Indicators, $\Delta n_{t,i} = 3$ months since each inflation forecast is for a future quarter of a year. However, starting in December of 1991, our data includes quarterly-frequency predictions by the Survey of Professional Forecasters for the inflation rate over the next 10 years. Thus, for the months of December, March, June, and September, we have an additional forecast where $\Delta n_{t,i} = 120$ - 0 =120 months.

Furthermore, let $k_t^c(n_i) \equiv \ln\left[1 + k^*\left(t; t - d\Delta t, t + (n_i - d)\Delta t\right)\right]$ be the continuously-compounded inflation swap rate whose payment date, $t_p$, is $n_i$ periods in the future (at date $t_p = t + n_i \Delta t$). Since $n_i \Delta t = t_e - t_s$, based on (23) it is assumed to take the form

$$\begin{aligned} k_t^c(n_i) &= \frac{1}{n_i \Delta t} \ln\left(\frac{V^d\left(t, t - d\Delta t, t + (n_i - d)\Delta t\right)}{P\left(t, t + n_i \Delta t\right)}\right) + \mu_{t,i} \\ &= y_t^n(n_i) - y_t^r(n_i) + \mu_{t,i} \end{aligned} \tag{26}$$

where $\mu_{t,i}$ is an independent measurement error distributed $N\left(0, u^2\right)$ and where $y_t^n(n_i)$ and $y_t^r(n_i)$ are the continuously-compounded yields on zero-coupon nominal bonds and real bonds (TIPS), respectively, that make their payments at date $t_p = t + n_i \Delta t$. Substituting in from (12) and (20), (26) can be re-written as

$$k_t^c(n_i) = \frac{1}{n_i \Delta t}\left[\Delta K_i^\dagger + \Delta A_i^\dagger \pi_t + \Delta B_i^\dagger r_t + \Delta C_i^\dagger \alpha_t + \textstyle\sum_{j=1}^4 \Delta D_{j,i}^\dagger h_{j,t}^2\right] + u_{t,i} \tag{27}$$

where $\Delta K_i^\dagger \equiv \ln\left(I_t / I_{t-d\Delta t}\right) + K_{n_i} - \tilde{K}_{n_i - d}$, $\Delta A_i^\dagger \equiv A_{n_i} - \tilde{A}_{n_i - d}$, $\Delta B_i^\dagger \equiv B_{n_i} - \tilde{B}_{n_i - d}$, $\Delta C_i^\dagger \equiv C_{n_i} - \tilde{C}_{n_i - d}$, and $\Delta D_{j,i}^\dagger \equiv D_{j,n_i} - \tilde{D}_{j,n_i - d}$.

While most bond yields and inflation forecasts are assumed to be observed with error, we need to assume perfect observation of the short term (one-month maturity) nominal rate, $i_t = y_t(1)$ $= -\frac{1}{\Delta t} \ln\left[P\left(t, t + \Delta t\right)\right] = \pi_t + r_t + \phi_1 h_{1,t}^2$, and the survey inflation forecast at the one-month horizon, $I\left(t, t + \Delta t\right) = \frac{1}{\Delta t} E_t\left[I_{t+\Delta t} / I_t\right] = \exp\left(\pi_t \Delta t\right)$. These assumptions allow us to recover the exact one period real rate, $r_t = i_t - \pi_t - \phi_1 h_{1,t}^2$, given that $h_{1,t}^2$ is observed. Unfortunately, knowledge of the exact values of the $i_t$, $\pi_t$, and $r_t$ by themselves, is not sufficient to update all of the volatility factors $h_{i,t}$, $i = 1, ..., 4$, because we also need to observe the central tendency, $\alpha_t$. Therefore, we also assume that another particular maturity nominal bond yield is measured without error. For, example, if this particular yield has maturity $n_x$, then with $\varepsilon_{t,x} = 0$ we have:

$$\begin{aligned} y_t(n_x) &= -\frac{1}{n_x \Delta t} \ln\left[P\left(t, t + n_x \Delta t\right)\right] \\ &= \frac{1}{n_x \Delta t}\left(K_{n_x} + A_{n_x} \pi_t + B_{n_x} r_t + C_{n_x} \alpha_t + \textstyle\sum_{j=1}^4 D_{j,n_x} h_{j,t}^2\right) \end{aligned} \tag{28}$$

which implies:

$$\alpha_t = \frac{1}{C_{n_x}}\left(n_x \Delta t y_t\left(n_x\right) - K_{n_x} - A_{n_x}\pi_t - B_{n_x} r_t - \sum_{j=1}^{4} D_{j,n_x} h_{j,t}^2\right) \tag{29}$$

In principle, the perfectly observed yield $y_t\left(n_j\right)$ could be chosen from any one of the available yields in our data sample. However, because this yield is used to identify the central tendency, $\alpha_t$, which largely determines the slope of the term structure, it would be reasonable to select a relatively long maturity bond yield. But since liquidity decreases as maturity expands, making the assumption of zero measurement error less plausible, as a compromise we initially select the five-year maturity ($n_x = 60$) as the maturity with no measurement error.

These assumptions allow us to observe $\pi_t, r_t$,and $\alpha_t$ and recover the $\epsilon_{j,t+\Delta t}$, $j = 1, ..., 4$ in equations (3) and (8). In turn, this allows us to update each of the volatility factors, $h_{j,t}$, $j = 1, ..., 4$. Given the state variables, $(\pi_t, r_t, \alpha_t, h_{j,t}^2, j = 1, .., 4)$ at date $t$, all the theoretical zero coupon bond yields, inflation forecasts, and inflation swap rates can be computed. The difference between these theoretical quantities and their actual counterparts determine the measurement errors for bond yields, inflation forecasts, and inflation swap rates. The likelihood function can then be calculated recursively.

Let $n_1$, ..., $n_b$ be the maturities of the $b$ different bonds whose yields are assumed to be measured with error, let $\left(n_{t,1}^b, n_{t,1}^e\right)$,..., $\left(n_{t,p}^b, n_{t,f}^e\right)$ be the horizons of the $f$ different inflation rate forecasts that are assumed to be measured with error at date $t$, and let $n_1$, ..., $n_p$ be the maturities of the $p$ different swap rates that are assumed to be measured with error. Note that at each monthly observation date, the bond yield maturities measured with error $n_1$, ..., $n_b$ are the same, equal to 3, 6, 12, 24, 36, 84, 120, and 180 months. However, due to the nature of the inflation survey data, the number of inflation forecasts, $f$, and their horizons vary over different observation months. Similarly, the number of inflation swap rates, $p$, (but not their horizons) vary over different observation months. However, for a given observation month and number of inflation forecasts, $f$, and swap rates, $p$, define

$$
Y_t = \begin{pmatrix} ln\left(\frac{I_{t+\Delta t}}{I_t}\right) \\ \pi_{t+\Delta t} \\ r_{t+\Delta t} \\ \alpha_{t+\Delta t} \\ y_t(n_1) \\ \vdots \\ y_t(n_b) \\ s_t\left(n^b_{t,1}, n^e_{t,1}\right) \\ \vdots \\ s_t\left(n^b_{t,f}, n^e_{t,f}\right) \\ k^c_t\left(n_1\right) \\ \vdots \\ k^c_t\left(n_p\right) \end{pmatrix} \quad A_t = \begin{pmatrix} 0 \\ 0 \\ b_0\Delta t \\ c_0\Delta t \\ \frac{K_{n_1}}{n_1\Delta t} \\ \vdots \\ \frac{K_{n_b}}{n_b\Delta t} \\ \frac{\Delta K^*_{t,1}}{\Delta n_{t,1}\Delta t} \\ \vdots \\ \frac{\Delta K^*_{t,f}}{\Delta n_{t,f}\Delta t} \\ \frac{\Delta K^\dagger_{n_1}}{n_1\Delta t} \\ \vdots \\ \frac{\Delta K^\dagger_{n_p}}{n_p\Delta t} \end{pmatrix} \tag{30}
$$

$$
X_t = \begin{pmatrix}
\Delta t & 0 & 0 & -\frac{1}{2}\Delta t & 0 & 0 & 0 \\
1+a_2\Delta t & a_1\Delta t & \Delta t & 0 & 0 & 0 & 0 \\
b_2\Delta t & 1+b_1\Delta t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1+c_1\Delta t & 0 & 0 & 0 & 0 \\
\frac{A_{n_1}}{n_1\Delta t} & \frac{B_{n_1}}{n_1\Delta t} & \frac{C_{n_1}}{n_1\Delta t} & \frac{D_{1,n_1}}{n_1\Delta t} & \frac{D_{2,n_1}}{n_1\Delta t} & \frac{D_{3,n_1}}{n_1\Delta t} & \frac{D_{4,n_1}}{n_1\Delta t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{A_{n_b}}{n_b\Delta t} & \frac{B_{n_b}}{n_b\Delta t} & \frac{C_{n_b}}{n_b\Delta t} & \frac{D_{1,n_b}}{n_b\Delta t} & \frac{D_{2,n_b}}{n_b\Delta t} & \frac{D_{3,n_b}}{n_b\Delta t} & \frac{D_{4,n_b}}{n_b\Delta t} \\
\frac{\Delta A^*_{t,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta B^*_{t,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta C^*_{t,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,1,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,2,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,3,1}}{\Delta n_{t,1}\Delta t} & \frac{\Delta D^*_{t,4,1}}{\Delta n_{t,1}\Delta t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\Delta A^*_{t,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta B^*_{t,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta C^*_{t,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,1,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,2,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,3,f}}{\Delta n_{t,f}\Delta t} & \frac{\Delta D^*_{t,4,f}}{\Delta n_{t,f}\Delta t} \\
\frac{\Delta A^\dagger_{n_1}}{n_1\Delta t} & \frac{\Delta B^\dagger_{n_1}}{n_1\Delta t} & \frac{\Delta C^\dagger_{n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{1,n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{2,n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{3,n_1}}{n_1\Delta t} & \frac{\Delta D^\dagger_{4,n_1}}{n_1\Delta t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\Delta A^\dagger_{n_p}}{n_p\Delta t} & \frac{\Delta B^\dagger_{n_p}}{n_p\Delta t} & \frac{\Delta C^\dagger_{n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{1,n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{2,n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{3,n_p}}{n_p\Delta t} & \frac{\Delta D^\dagger_{4,n_p}}{n_p\Delta t}
\end{pmatrix} \tag{31}
$$

Then, our system of equations to be estimated can be written

$$
Y_t = A_t + X_t B_t + \Psi_t \tag{32}
$$

where

$$B_t = \begin{pmatrix} \pi_t \\ r_t \\ \alpha_t \\ h_{1,t}^2 \\ h_{2,t}^2 \\ h_{3,t}^2 \\ h_{4,t}^2 \end{pmatrix}, \Psi_t = \begin{pmatrix} \sqrt{\Delta t} h_{1,t}\epsilon_{1,t+\Delta t} \\ \sqrt{\Delta t}\sum_{j=1}^{2}\beta_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \sqrt{\Delta t}\sum_{j=1}^{3}\gamma_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \sqrt{\Delta t}\sum_{j=1}^{4}\rho_j h_{j,t}\epsilon_{j,t+\Delta t} \\ \omega_{t,1} \\ \vdots \\ \omega_{t,b} \\ \upsilon_{t,1} \\ \vdots \\ \upsilon_{t,f} \\ \mu_{t,1} \\ \vdots \\ \mu_{t,p} \end{pmatrix}. \tag{33}$$

The $\omega_{t,i}$ for $i = 1,..,b$ , the $\upsilon_{t,i}$ for $i = 1,..,f$ , and the $\mu_{t,i}$ for $i = 1,...,p$ are a sequence of independent normally distributed measurement errors.

Let $\Sigma_t$ represent the variance covariance matrix of $\Psi_t$. It has the block diagonal form:

$$\Sigma_t = \begin{pmatrix} \Delta t H_t & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & U \end{pmatrix} \tag{34}$$

where

$$H_t = \begin{pmatrix} h_{1,t}^2 & \beta_1 h_{1,t}^2 & \gamma_1 h_{1,t}^2 & \rho_1 h_{1,t}^2 \\ \beta_1 h_{1,t}^2 & \sum_{j=1}^{2}\beta_j^2 h_{j,t}^2 & \sum_{j=1}^{2}\beta_j\gamma_j h_{j,t}^2 & \sum_{j=1}^{2}\beta_j\rho_j h_{j,t}^2 \\ \gamma_1 h_{1,t}^2 & \sum_{j=1}^{2}\beta_j\gamma_j h_{j,t}^2 & \sum_{j=1}^{3}\gamma_j^2 h_{j,t}^2 & \sum_{j=1}^{3}\gamma_j\rho_j h_{j,t}^2 \\ \rho_1 h_{1,t}^2 & \sum_{j=1}^{2}\beta_j\rho_j h_{j,t}^2 & \sum_{j=1}^{3}\gamma_j\rho_j h_{j,t}^2 & \sum_{j=1}^{4}\rho_j^2 h_{j,t}^2 \end{pmatrix} \tag{35}$$

and where $W = w^2 I_b$, $V = v^2 I_f$, $U = u^2 I_p$, and $I_b$, $I_f$, and $I_p$ are $b \times b$, $f \times f$ , $p \times p$ identity matrices, respectively.

In principle, the model's 36 parameters can be estimated in one step using equation (32). However, note that the first element of $Y_t$ is the process for the log of actual inflation, $\ln(I_{t+\Delta t}/I_t) = \pi_t \Delta t - \frac{1}{2}\Delta t h_{1,t}^2 + \sqrt{\Delta t} h_{1,t}\epsilon_{1,t+\Delta t}$. By estimating this equation alone using data only on $I_t$ and $\pi_t$, we can recover estimates of the four parameters of the $h_{1,t}$ GARCH process, namely $d_{10}$ (equivalently, $\overline{h}_1$), $d_{11}$, $d_{12}$, and $d_{13}$. Therefore, to make overall parameter estimation more manageable, we implement a two-step procedure where we first estimate the parameters of the $h_{1,t}$ process using data only on $I_t$ and $\pi_t$. The 32 other parameters are estimated in a second step using equation (32) but with the parameters of the $h_{1,t}$ process fixed at those estimated in the first step. This two-step procedure would be equivalent to the one step maximum likelihood procedure but where the first set of observations on the log of actual inflation, $\ln(I_{t+\Delta t}/I_t)$, is given a much larger weight than the other observations.

# 5 Empirical Results

Table 1 presents results of the first step estimation of the parameters of the inflation volatility process $h_{1,t}$ using data on the CPI ($I_t$) and the one-month Blue Chip forecast of inflation ($\pi_t$) over the period January 1982 to June 2008. The annualized, conditional standard deviation for inflation over a one-month horizon has a steady-state value of $\overline{h}_1 = 0.0083$; that is, 83 basis points. The volatility of inflation displays significant GARCH effects: the estimated coefficient on the shock to inflation in the GARCH updating, $d_{12}$, is significantly positive. However, since $d_{13}$ is insignificantly different from zero, there is no evidence that inflation's volatility responds asymmetrically to surprises.

Table 2 reports second step estimates of the model's other parameters. To gauge the statistical significance of permitting GARCH behavior, we estimated the unrestricted model as well as restricted models that assumed some of the volatilities were constant; that is, $h_{j,t} = \overline{h}_j$. An assumption of constant volatility for a process $h_{j,t}$ entails the restrictions $d_{j1} = -1/\Delta t = -12$ and $d_{j2} = d_{j3} = 0$. The first column of Table 2 reports estimates assuming no GARCH behavior ($h_{j,t} = \overline{h}_j$, for $j = 2$, 3, and 4); the second column assumes GARCH behavior for only $h_{2,t}$; the third column assumes GARCH behavior only for $h_{3,t}$; and the fourth column assumes GARCH behavior for only $h_{4,t}$. Finally, the last column of Table 2 is the unrestricted model that permits GARCH behavior for $h_{2,t}$, $h_{3,t}$, and $h_{4,t}$.

Inspection of the log likelihood values for the different models at the bottom of Table 2 indicates that one can reject at the 99% confidence level the hypothesis of no GARCH behavior for all but one of the less restricted cases. The largest increase in likelihood value from loosing the no GARCH assumption for any single volatility process occurs for $h_{3,t}$, the volatility process for the independent component of the real interest rate, $r_t$. The second largest increase occurs when $h_{2,t}$ is able to display GARCH, which is the independent volatility component for expected inflation, $\pi_t$. The only instance where no GARCH behavior cannot be rejected occurs when only $h_{4,t}$, the independent volatility component of the central tendency, is permitted to display GARCH. However, as indicated in the last column of Table 2, when $h_{2,t}$, $h_{3,t}$, and $h_{4,t}$ are all allowed to follow GARCH processes, one can reject the no GARCH restriction and each of the GARCH parameters ($d_{22}$, $d_{32}$, and $d_{42}$) are significantly positive.

It is noteworthy that the estimates imply reasonable values for the unconditional means of inflation and the real interest rate. For example, the unrestricted model implies a value for $\overline{\pi}$ of 3.22% and for $\overline{r}$ of 1.57%. In terms of the model's fit to the data, the estimated standard deviations of measurement errors for Treasury yields, survey forecasts of inflation rates, and inflation swap rates ($w$, $v$, and $u$) are 35 basis points, 39 basis points, and 27 basis points, respectively. Given the unrestricted model's parameter estimates, we can also calculate the model's implied standard deviations and correlations for inflation, expected inflation, the real rate, and the central tendency based on these variables' covariance matrix, $H_t$, given in equation (35). This is done in Table 3 where each of the GARCH processes are assumed to equal their steady state values; that is, $h_{j,t} = \overline{h}_j$, $j = 1, ..., 4$.

The diagonal in Table 3 reports the state variables' annualized, steady-state standard deviations over a one-month horizon. The real interest rate, $r_t$, and expected inflation, $\pi_t$, have the highest unconditional standard deviations of 3.18% and 2.66%, respectively. Conditional on its mean of $\pi_t$, the steady-state one-month standard deviation of log inflation is 0.83% while the steady-state standard deviation of the central tendency is 1.04%. Examining the off-diagonal elements in Table 3, one see that inflation and changes in expected inflation display an average correlation of 0.33. Consistent with previous research (Pennacchi 1991), we also find strong negative correlation between expected inflation and the real interest rate of -0.844.

Of course due to GARCH behavior, the standard deviations of the state variables are not constant. Figure 1 displays the time series of the standard deviations of $\ln(I_{t+\Delta t}/I_t)$, $\pi_t$, $r_t$, and $\alpha_t$. As one would expect, the standard deviations of expected inflation and the real interest rate were especially high during the early 1980s, a time when the Federal Reserve was battling to lower inflation expectations. Confirming the estimation results in Table 2, this figure also shows that there is little evidence of GARCH behavior for the central tendency.

In contrast to their standard deviations, Figure 2 plots the model-implied levels of the state variables over the 1982 to 2008 period. The first panel in the figure indicates that the Federal Reserve was successful in lowering expected inflation. It shows that early in the period the central tendency for inflation was above short-run expected inflation as investors apparently thought medium-term inflation was likely to remain high. However, the Federal Reserve appears to have built credibility in lowering inflation, since the central tendency later declined to equal approximately the average of expected inflation. Beginning in 2008, the model is predicting that both expected inflation and its central tendency are on the rise.

The second panel in Figure 2 displays the real interest rate, $r_t$, implied by our model estimates. There was an unusually long period from 2002 to 2005 when it was negative. This finding supports those who believe that a credit bubble may have been inflated by a policy of maintaining interest rates too low for too long. The figure also shows that currently the short-run real interest rate is quite negative.

Figures 3 and 4 characterize the speeds of mean reversion for the state variables $\pi_t$, $r_t$, and $\alpha_t$. Figure 3 presents impulse response functions that assume when there is a positive one-standard deviation shock to a state variable, there is no instantaneous shock to the other state variables. It shows that there is somewhat stronger mean reversion for expected inflation compared to the real interest rate. A shock to the central tendency displays very weak reversion to its mean. Figure 4 differs from Figure 3 in that when there is a positive one-standard deviation shock to a state variable, the other state variables also suffer a shock commensurate with the estimated correlations given in Table 3. Under this scenario, mean reversion for expected inflation and the real interest rate becomes somewhat stronger than before. However, allowing for contemporaneous state variable shocks has little effect on the weak mean reversion of the central tendency.

A basic question is whether our estimated model produces sensible-looking nominal and real (inflation-indexed) yield curves. Figure 5 shows the unrestricted model's implied yield curves and expected inflation when each of the state variables is initially at its steady state level ($\pi_t = \widehat{\alpha}_t = \overline{\pi}$, $r_t = \overline{r}$, $h_{j,t} = \overline{h}_j$, $j = 1, ..., 4$). Indeed, the term structures do appear reasonable, even for maturities out to 30 years, a horizon where little data was used in the model's estimation. The slopes of the steady-state nominal yield curve (difference between yields and the one-month nominal rate $i_t$) equal 137, 192, 236, and 245 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. Similarly, the slopes of the real yield curve (difference between the yields and the one-month real rate $r_t$) equal 109, 150, 192, and 213 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. The substantial slope of the real yield curve constrasts with other studies that find it to be relatively flat.[14]

A component of the nominal term structure that has interested policymakers and academics is the inflation risk premium. There are at least two motivations for wanting to know this quantity. First, saving the cost of the inflation risk premium has been used to justify a government's issuance of inflation-indexed bonds. Second, one needs to subtract the inflation risk premium from the "break-even inflation rate" (difference between equivalent maturity nominal and inflation-indexed bonds) in order to construct a measure of inflation expectations.

We quantify the inflation risk premium, as well as the real interest rate risk premium, in the following manner. First, we compute nominal and real yields under the assumption that all of the market prices of risk equal zero; that is, $\phi_j = 0$, $j = 1, ..., 4$. As an illustration, the zero-risk premia yield curves when all of the state variables are initially at their steady states are plotted in Figure 6. Second, the real interest rate risk premium then is defined as the difference between the real yield curve (with market prices of risk at their estimated values) minus the zero-risk premia real yield curve. This real risk premium is plotted in Figure 7, which is the difference between the real yield curves in Figures 5 and 6.

[14] See, for example, (Buraschi and Jiltsov 2005).

Third, we do a similar exercise for the nominal yield curve. Also plotted in Figure 7 is the nominal risk premium. It is defined as the difference between the nominal yield curve in Figure 5 (with market prices of risk at their estimated values) minus the zero-risk premia nominal yield curve given in Figure 6. Finally, the inflation risk premium is defined as the difference between the nominal risk premium and the real risk premium for the same maturity. Figure 7 also displays this inflation risk premium.

When each of the state variables are initially at their steady states, we see that the real risk premium equals 111, 156, 212, and 250 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. The inflation risk premium equals 27, 51, 82, and 101 basis points at the 5-, 10-, 20-, and 30-year maturities, respectively. We can also examine how these risk premia varied over time during our model's sample period. Figure 8 plots expected inflation, the real risk premium, and the inflation risk premium for the 10-year maturity during the 1982 to 2008 period. Interestingly, while inflation expected over the next 10 years varied substantially, the levels of the real and inflation risk premia did not. The real risk premium for a ten-year maturity bond varied from 150 to 170 basis points, averaging 157 basis points. This real risk premium appears significantly higher than that found in other studies, such as (Ang, Bekaert, and Wei 2007b). The ten-year inflation risk-premium for a ten-year maturity bond varied from 38 to 60 basis points and averaged 51 basis points. These estimates of the ten-year inflation risk premium fall within the range of those estimated by other studies.[15]

In the spirit of an out-of-sample test, we relate our model's implied yields for zero coupon inflation-indexed bonds to yields of actual zero-coupon TIPS. Recently, (Gurkaynak, Sack, and Wright 2008) have fit zero coupon TIPS yield curves based on the yields of actual coupon-paying TIPS.[16] Their data spans the period January 1999 to June 2008. Taking their 5- and 10-year zero-coupon TIPS yields, we compare them to our unrestricted model's implied 5- and 10-year zero-coupon yields for inflation-indexed bonds. The results of this exercise are given in Figures 9 and 10. These figures show that our model significantly overvalues both the 5- and 10-year TIPS until about 2004. However, during the last four years, our model's yields and the TIPS yields appear to be tightly linked. One interpretation of this comparison is that our model performs poorly in pricing inflation-indexed bonds during the 1999 to 2004 period. However, based on prior studies such as (Sack and Elsasser 2004) and (D'Amico, Kim, and Wei 2008), a more likely interpretation may be that TIPS were significantly undervalued prior to 2004. For example, (D'Amico, Kim, and Wei 2008) find a large "liquidity premium" during the early years of TIPS's existence, especially before 2002. They conclude that until more recently, TIPS yields were difficult to account for within a rational pricing framework. Hence, the overall evidence

[15]For example, a ten-year inflation risk premium averaging 70 basis points and ranging from 20 to 140 basis points is found by (Buraschi and Jiltsov 2005). Using data on TIPS, (Adrian and Wu 2008) find a smaller 10-year inflation risk premium varying between -20 and 20 basis points.

[16]Their dataset is available at http://www.federalreserve.gov/pubs/feds/2008/index.html.

points toward our model being able to generate fair prices for inflation-indexed bonds.

## 6 Conclusions

This paper presented an equilibrium model of the term structures of nominal and real interest rates. Its factors include the short-term real interest rate, the short-term expected inflation rate, and the inflation rate's central tendency. Along with actual inflation, these factors are assumed to be driven by four volatility processes that follow the nonlinear asymmetric GARCH model of Engle and Ng (Engle and Ng 1993). By allowing for a changing central tendency for inflation and for changing volatilities for real rates and inflation, our model is able to account for changing monetary and real economic conditions.

Although our model permits state variables to have a general correlation structure with stochastic volatilities, it still allows for analytical solutions for the prices of nominal bonds and inflation-indexed bonds that have an indexation lag, such as TIPS. Closed-form solutions for expected inflation rates and equilibrium rates on inflation swaps also can be derived.

The model's parameters were estimated using data on nominal Treasury yields, survey forecasts of inflation, and inflation swap rates. We found that allowing for GARCH effects is particularly important for real interest rate and expected inflation processes, but that long-horizon real and inflation risk premia are relatively stable. Our estimate for the ten-year inflation risk premium averaged 51 basis points and varied between 38 and 60 basis points during the 1982 to 2008 sample period. Somewhat different from prior studies, we find a sizeable real interest rate risk premium at the ten-year maturity, averaging 157 basis points and varying between 150 and 170 basis points.

Comparing our model's implied yields of inflation-indexed bonds to those of TIPS suggests that TIPS were underpriced prior to 2004 but more recently are fairly priced. Hence, the "liquidity premium" in TIPS yields appears to have disappeared. Perhaps the recent introduction of inflation derivatives, such as zero coupon inflation swaps, has helped to eliminate this premium by creating a more complete market for inflation-linked securities.

## Appendix

### Lemma 1

Let $X$ be a standard normal random variable. Then

$$E\left[e^{Q_1X-Q_2X^2}\right]=\frac{e^{Q_1^2}}{2(1+2Q_2)}-\frac{1}{2}ln(1+2Q_2) \tag{A.1}$$

**Proof:**

The expectation can be written as:

$$E\left[e^{Q_1X-Q_2X^2}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{Q_1x-Q_2x^2-\frac{1}{2}x^2}dx \tag{A.2}$$

The result follows after completing the square and using properties of the normal density function.

### Proof of Proposition 1

We begin by substituting the nominal pricing kernel into the bond pricing equation, (12). This leads to:

$$P(t,t+n\Delta t)=E_t\left[e^{-(\pi_t+r_t+\phi_1h_t^2)\Delta t-\frac{1}{2}\sum_{j=1}^4\phi_j^2h_{j,t}^2\Delta t+\sum_{j=1}^4\phi_jh_{j,t}\sqrt{\Delta t}\epsilon_{j,t+\Delta t}}P(t+\Delta t,t+n\Delta t)\right] \tag{A.3}$$

Now assume the bond price equation has the form:

$$P(t,t+n\Delta t)=e^{-(K_n+A_n\pi_t+B_nr_t+C_n\alpha_t+\sum_{j=1}^4D_{j,n}h_{j,t}^2)} \tag{A.4}$$

Then using this expression for $P(t+\Delta t,t+n\Delta t)$, in the bond pricing equation and gathering together all coefficients of the same random variables, and computing expectations using the lemma eventually leads to the final result. The initial boundary conditions come from considering the case when $n=1$.

### Proof of Proposition 2

Assume the structure for growth in the price level, $I(t,t+n\Delta t)$ has the following form:

$$I(t,t+n\Delta t)=e^{\bar{K}_n+\bar{A}_n\pi_t+\bar{B}_nr_t+\bar{C}_n\alpha_t+\sum_{j=1}^4\bar{D}_{j,n}h_{j,t}^2} \tag{A.5}$$

Then:

$$I(t,t+n\Delta t)=E_t\left[\frac{I_{t+\Delta t}}{I_t}I(t+\Delta t,t+n\Delta t)\right] \tag{A.6}$$

Substituting in the assumed form for $I(t+\Delta t, t+n\Delta t)$, we obtain

$$I(t, t+n\Delta t) = E_t\left[e^{\pi_t\Delta t-\frac{1}{2}h_t^2\Delta t+h_t\sqrt{\Delta t}\epsilon_{1,t+\Delta t}+\bar{F}_{n-1}+\bar{A}_{n-1}\pi_{t+\Delta t}+\bar{B}_{n-1}r_{t+\Delta t}+\bar{C}_{n-1}\alpha_{t+\Delta t}+\sum_{j=1}^{4}\bar{D}_{j,n-1}h_{j,t+\Delta t}^2}\right]. \tag{A.7}$$

Substituting in the dynamics for the state variables, and collecting all coefficients of the random variables of the same type together, and then taking expectations leads, after tedious computations, to the resulting recursion. The initial conditions come from the case when considering $n = 1$.

**Proof of Proposition 3**

Assume the following structure:

$$V^d(t, t+n\Delta t) = \frac{I_t}{I_{0-d\Delta t}}e^{-\tilde{K}_n-\tilde{A}_n\pi_t-\tilde{B}_nr_t-\tilde{C}_n\alpha_t-\sum_{j=1}^{4}\tilde{D}_{j,n}h_{j,t}^2} \tag{A.8}$$

Now

$$V^d(t, t+n\Delta t) = E_t\left[\frac{M_{t+\Delta t}}{M_t}V^d(t+\Delta t, t+n\Delta t)\right] \tag{A.9}$$

Substituting in the structure for $V^d(t+\Delta t, t+n\Delta t)$ leads to:

$$V^d(t, t+n\Delta t) = E_t\left[\frac{I_{t+\Delta t}}{I_{0-d\Delta t}}\frac{M_{t+\Delta t}}{M_t}e^{-\tilde{K}_{n-1}-\tilde{A}_{n-1}\pi_{t+\Delta t}-\tilde{B}_{n-1}r_{t+\Delta t}-\tilde{C}_{n-1}\alpha_{t+\Delta t}-\sum_{j=1}^{4}\tilde{D}_{j,n-1}h_{j,t+\Delta t}^2}\right] \tag{A.10}$$

This can be rewritten as:

$$V^d(t, t+n\Delta t) = \frac{I_t}{I_{0-d\Delta t}}E_t\left[\frac{I_{t+\Delta t}}{I_t}\frac{M_{t+\Delta t}}{M_t}e^{-\tilde{K}_{n-1}-\tilde{A}_{n-1}\pi_{t+\Delta t}-\tilde{B}_{n-1}r_{t+\Delta t}-\tilde{C}_{n-1}\alpha_{t+\Delta t}-\sum_{j=1}^{4}\tilde{D}_{j,n-1}h_{j,t+\Delta t}^2}\right] \tag{A.11}$$

The recursive equations follow by substituting in the dynamics for the pricing kernel, inflation, and the expressions for the state variables, gathering all common terms together for the coefficients of each random variables, $\epsilon_{j,t+\Delta t}$ and $\epsilon_{j,t+\Delta t}^2$ for $j = 1, .., 4$, computing the expectations using the above lemma, and rearranging the resulting terms together.

The boundary conditions are obtained by recognizing that at date $t+n\Delta t$, the final payment is known, but is deferred by $d$ periods. So the boundary conditions with no periods to go are given by the known payment multiplied by the $d$-period discount bond price, the formula for which is given in Proposition 1.

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**Table 1**

**Estimates of the GARCH Process for Inflation**

$$\frac{I_{t+\Delta t}}{I_t} = e^{\pi_t \Delta t - \frac{1}{2} h_{1,t}^2 \Delta t + h_{1,t} \sqrt{\Delta t} \varepsilon_{1,t+\Delta t}}$$

$$h_{1,t+\Delta t}^2 - h_{1,t}^2 = \left[ d_{10} + d_{11} h_{1,t}^2 + d_{12} \left( \varepsilon_{1,t+\Delta t} - d_{13} h_{1,t} \right)^2 \right] \Delta t$$

$$\bar{h}_1^2 = -\frac{d_{10} + d_{12}}{d_{11} + d_{12} d_{13}^2}$$

| **Parameter** | **Estimate** | ***t*-Statistic** | ***p*-Value** |
|---|---|---|---|
| $\bar{h}_1$ | 0.0083 | 3.98 | 0.000 |
| $d_{11}$ | -1.446 | -1.86 | 0.063 |
| $d_{12}$ | $1.23 \times 10^{-4}$ | 2.34 | 0.019 |
| $d_{13}$ | -5.86 | -0.19 | 0.849 |
| | | | |
| Observations | 318 | | |

**Table 2**

**Real and Nominal Term Structure Parameter Estimates**

$$\pi_{t+\Delta t} - \pi_t = [\alpha_t + a_1 r_t + a_2 \pi_t]\Delta t + \sqrt{\Delta t}\ \sum_{j=1}^{2} \beta_j h_{j,t} \varepsilon_{j,t+\Delta t}$$

$$r_{t+\Delta t} - r_t = [b_0 + b_1 r_t + b_2 \pi_t]\Delta t + \sqrt{\Delta t}\ \sum_{j=1}^{3} \gamma_j h_{j,t} \varepsilon_{j,t+\Delta t}$$

$$\alpha_{t+\Delta t} - \alpha_t = [c_0 + c_1 \alpha_t]\Delta t + \sqrt{\Delta t}\ \sum_{j=1}^{4} \rho_j h_{j,t} \varepsilon_{j,t+\Delta t}$$

$$h_{j,t+\Delta t}^2 - h_{j,t}^2 = \left[ d_{j0} + d_{j1} h_{j,t}^2 + d_{j2} \left( \varepsilon_{j,t+\Delta t} - d_{j3} h_{j,t} \right)^2 \right]\Delta t$$

$$\bar{\pi} = -\frac{a_1 b_0 c_1 + b_1 c_0}{c_1 \left( a_1 b_2 - a_2 b_1 \right)}, \quad \bar{r} = -\frac{a_2 b_0 c_1 + b_2 c_0}{c_1 \left( a_1 b_2 - a_2 b_1 \right)}$$

$$\bar{h}_j^2 = -\frac{d_{j0} + d_{j2}}{d_{j1} + d_{j2} d_{j3}^2}, \quad \text{Risk premia: } \phi_j h_{j,t}, \quad j = 1,2,3,4$$

| Parameter | No GARCH | $h_2$ GARCH | $h_3$ GARCH | $h_4$ GARCH | $h_2$, $h_3$, $h_4$ GARCH |
|---|---|---|---|---|---|
| $\bar{\pi}$ | 0.0258*** | 0.0241*** | 0.0416*** | 0.0258*** | 0.0322*** |
| $a_1$ | 0.6929*** | 0.5625*** | 0.4949*** | 0.6928*** | 0.4916*** |
| $a_2$ | -2.762*** | -2.7236*** | -2.3683*** | -2.762*** | -2.4224*** |
| $\beta_1$ | 0.7405*** | 0.8285*** | 0.8323*** | 0.7415*** | 1.0669*** |
| $b_1$ | -1.8036*** | -1.4668*** | -1.2476*** | -1.8027*** | -1.2543*** |
| $b_2$ | 2.7787*** | 2.2795*** | 1.848*** | 2.7773*** | 1.8482*** |
| $\gamma_1$ | -0.1311 | -0.597*** | -0.6659*** | -0.131 | -0.9093*** |
| $\gamma_2$ | -0.9822*** | -1.3765*** | -1.0281*** | -0.9826*** | -1.0295*** |
| $\bar{r}$ | 0.0126** | 0.0119** | 0.0252*** | 0.0127** | 0.0157** |
| $c_1$ | -0.0541*** | -0.051*** | -0.0482*** | -0.0541*** | -0.0483*** |
| $\rho_1$ | -0.0427 | -0.0188 | 0.1372** | -0.0418 | 0.1975*** |
| $\rho_2$ | -0.0242 | -0.0365*** | 0.017 | -0.0278 | 0.0346* |
| $\rho_3$ | -0.0752*** | -0.073*** | -0.1408*** | -0.0759*** | -0.1586*** |
| $\bar{h}_2$ | 0.02494*** | 0.02504*** | 0.024576*** | 0.02494*** | 0.02508*** |
| $d_{21}$ | | -1.9082*** | | | -1.9891*** |
| $d_{22}$ | | 0.001388*** | | | 0.001419*** |
| $d_{23}$ | | -25.48*** | | | 0.08 |
| $\bar{h}_3$ | 0.02571*** | 0.026851*** | 0.016703*** | 0.02571*** | 0.017029*** |
| $d_{31}$ | | | -6.033*** | | -6.0448*** |
| $d_{32}$ | | | 0.001331*** | | 0.001386*** |
| $d_{33}$ | | | 33.33*** | | 32.61*** |
| $\bar{h}_4$ | 0.010536*** | 0.011916*** | 0.010296*** | 0.010488*** | 0.009849*** |
| $d_{41}$ | | | | -11.9984 | -2.2841*** |
| $d_{42}$ | | | | 0.000022 | 0.000031*** |
| $d_{43}$ | | | | -0.51 | -179.51*** |
| $\phi_1$ | 60.06*** | 58.2*** | -16.00*** | 59.96*** | -19.93*** |
| $\phi_2$ | -39.54*** | -39.35*** | -4.06** | -39.49*** | 6.88*** |
| $\phi_3$ | -7.41 | -13.17*** | 50.18*** | -7.44 | 50.25*** |
| $\phi_4$ | 19.27*** | 13.25*** | 27.54*** | 18.40*** | 36.27*** |

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| *w* | 0.0038 | 0.0037 | 0.0035 | 0.0038 | 0.0035 |
| *v* | 0.0039 | 0.0038 | 0.0040 | 0.0039 | 0.0039 |
| *u* | 0.0027 | 0.0026 | 0.0027 | 0.0027 | 0.0027 |
| | | | | | |
| Ln Likelihood | 32443.314 | 32515.500 | 32632.206 | 32443.314 | 32686.266 |
| Reject No GARCH? | | Yes | Yes | No | Yes |

Note: ***, **, and * denotes statistical significance at the 1%, 5%, and 10% level. *w*, *v*, and *u* are the standard deviations of the measurement errors for nominal Treasury yields, survey inflation rate forecasts, and inflation swap rates, respectively. For each set of estimates, the parameters of the GARCH process for inflation ($h_1$) are fixed at the point estimates reported in Table 1.

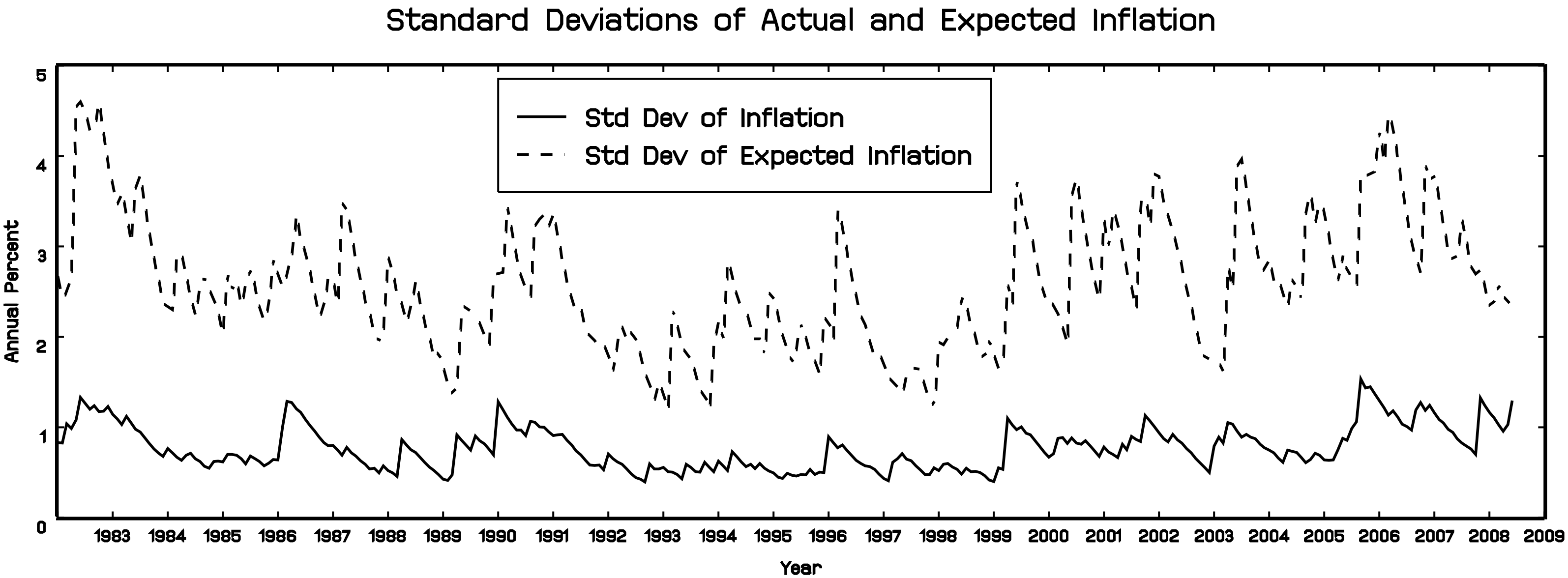
**Table 3**

**Estimated Steady State Standard Deviations and Correlations**

| | | | | |
|---|---|---|---|---|
| | $\ln(I_{t+\Delta t}/I_t)$ | $\pi_{t+\Delta t}$ | $r_{t+\Delta t}$ | $\alpha_{t+\Delta t}$ |
| $\ln(I_{t+\Delta t}/I_t)$ | 0.0083 | | | |
| $\pi_{t+\Delta t}$ | 0.333 | 0.0266 | | |
| $r_{t+\Delta t}$ | -0.237 | -0.844 | 0.0318 | |
| $\alpha_{t+\Delta t}$ | 0.158 | 0.131 | -0.244 | 0.0104 |

Note: The diagonal elements are annualized, one-month standard deviations while the off-diagonal elements are one-month correlations. These standard deviations and correlations assume that the GARCH processes are at their steady-state levels: $h_{j,t} = \bar{h}_j$, $j$ = 1,…,4. Parameter estimates are from the full model.

Figure 1


Standard Deviations of Actual and Expected Inflation
Std Dev of Inflation
Std Dev of Expected Inflation
Annual Percent
Year
0
1
2
3
4
5
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
1995
1996
1997
1998
1999
2000
2001
2002
2003
2004
2005
2006
2007
2008
2009

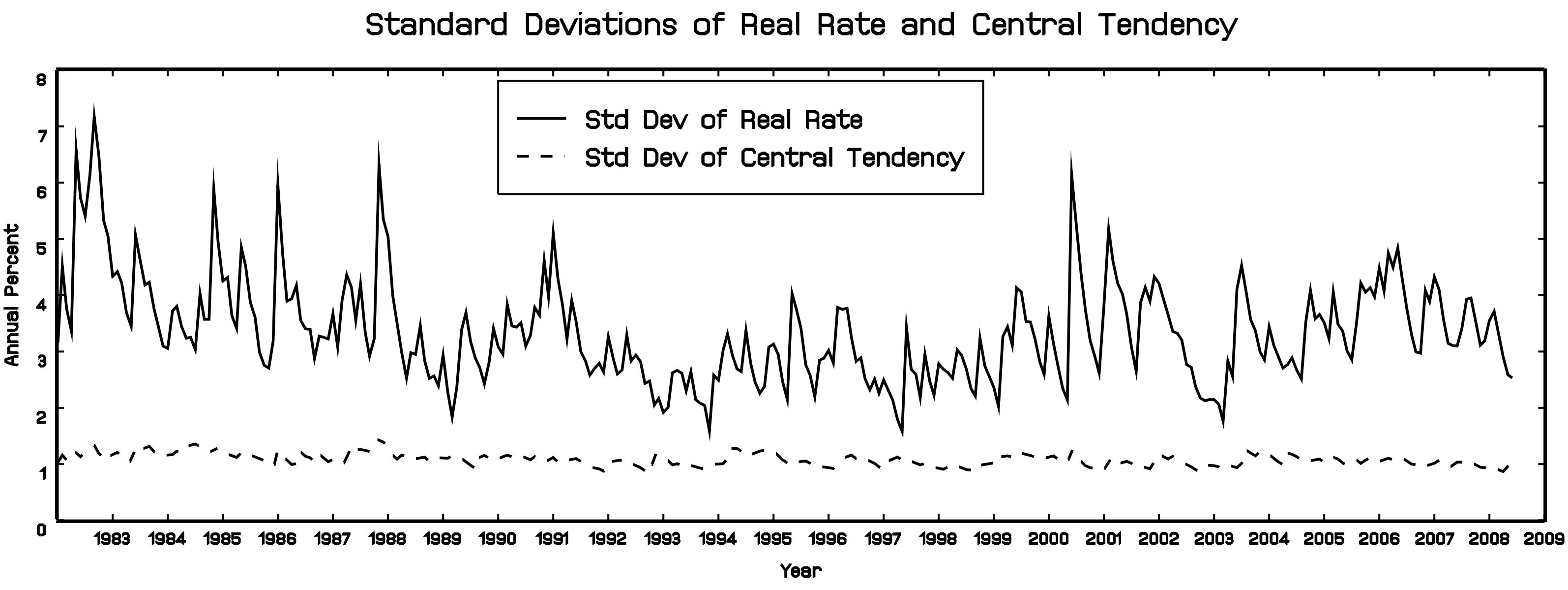
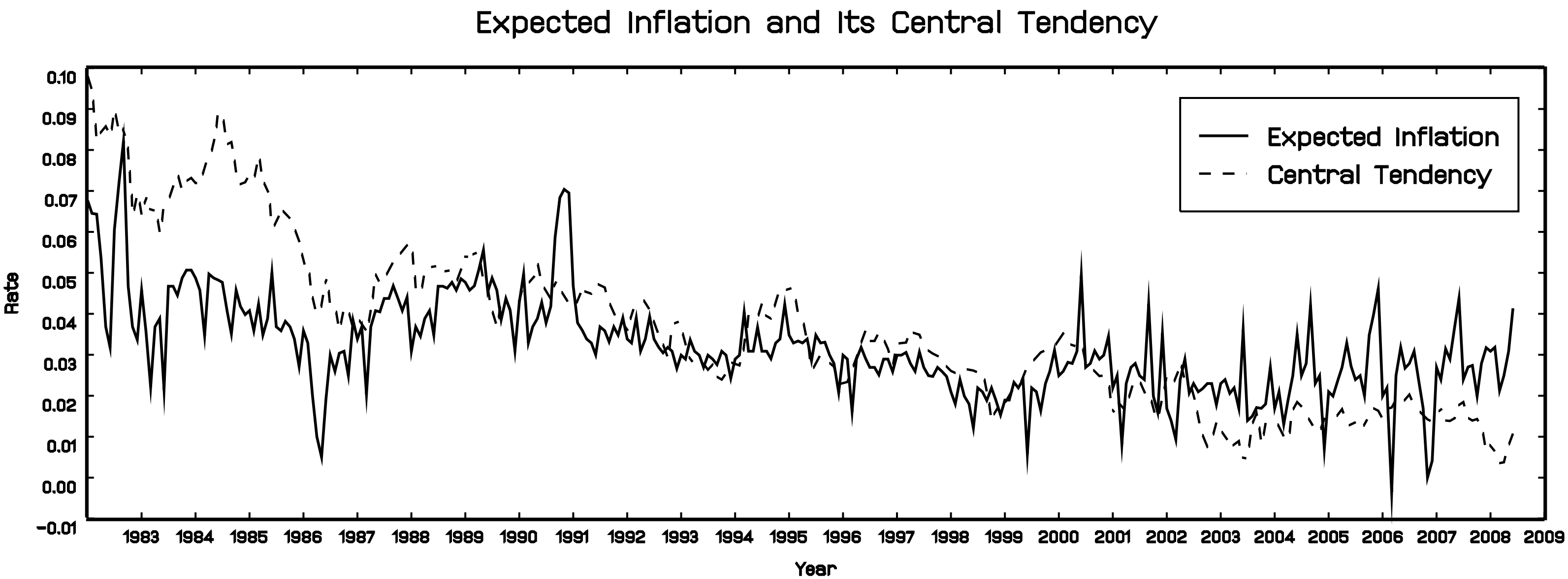

Standard Deviations of Real Rate and Central Tendency
Std Dev of Real Rate
Std Dev of Central Tendency
Annual Percent
Year
0
1
2
3
4
5
6
7
8
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
1995
1996
1997
1998
1999
2000
2001
2002
2003
2004
2005
2006
2007
2008
2009

Figure 2


Expected Inflation and Its Central Tendency
Expected Inflation
Central Tendency
Rate
Year
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
-0.01
1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009

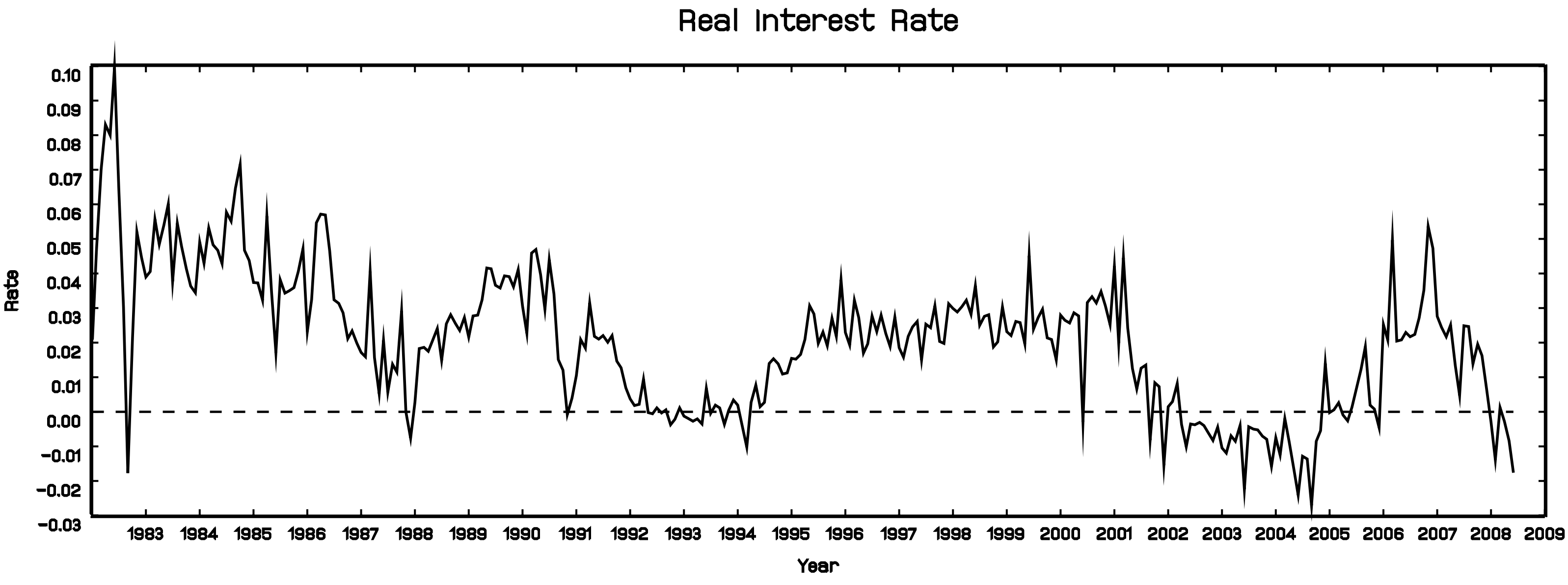

Real Interest Rate
Rate
Year
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
-0.03
1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009

Figure 3

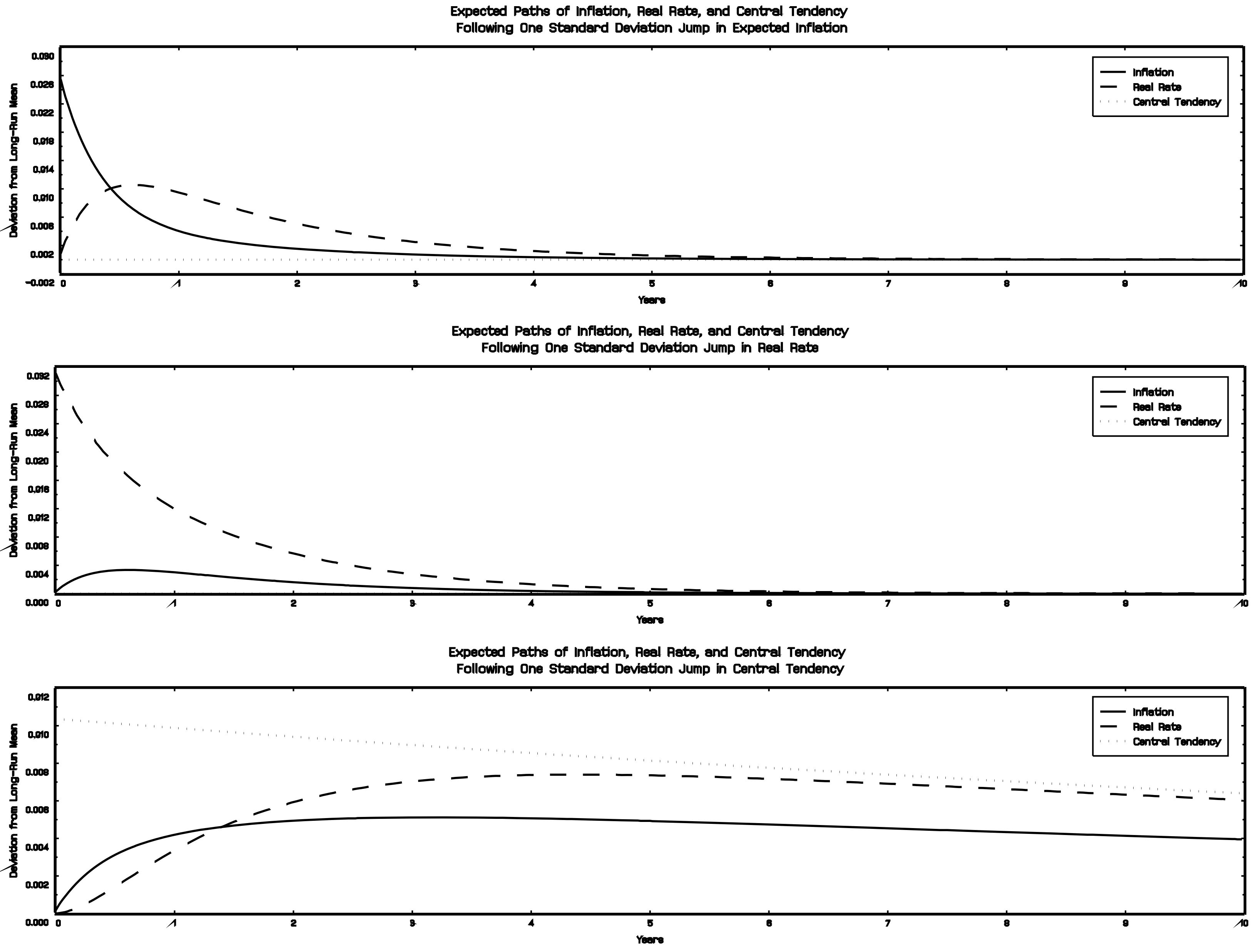

Expected Paths of Inflation, Real Rate, and Central Tendency
Following One Standard Deviation Jump in Expected Inflation
Deviation from Long-Run Mean
Years
Inflation
Real Rate
Central Tendency
Expected Paths of Inflation, Real Rate, and Central Tendency
Following One Standard Deviation Jump in Real Rate
Deviation from Long-Run Mean
Years
Inflation
Real Rate
Central Tendency
Expected Paths of Inflation, Real Rate, and Central Tendency
Following One Standard Deviation Jump in Central Tendency
Deviation from Long-Run Mean
Years
Inflation
Real Rate
Central Tendency

Figure 4

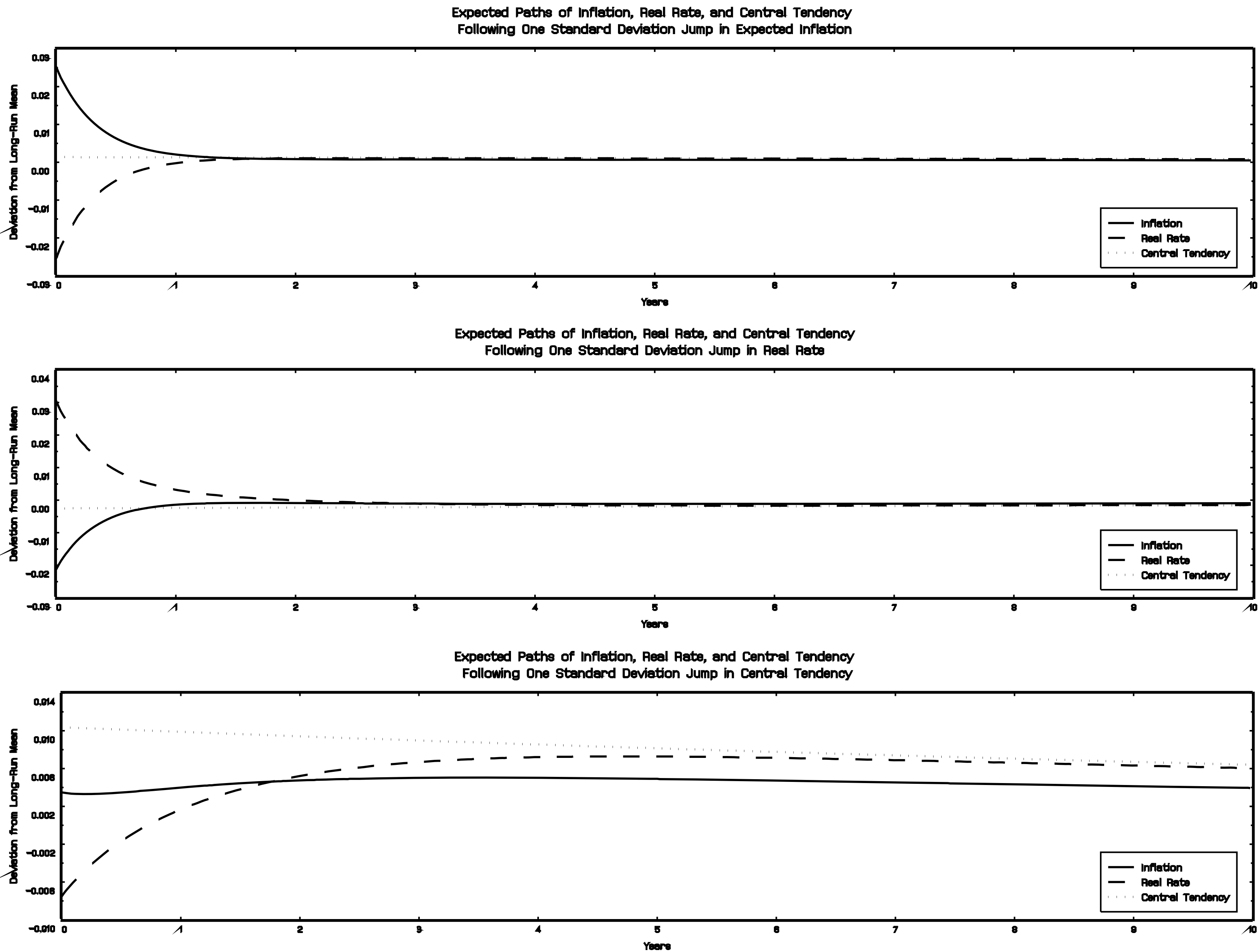

Expected Paths of Inflation, Real Rate, and Central Tendency
Following One Standard Deviation Jump in Expected Inflation
Deviation from Long-Run Mean
Years
Inflation
Real Rate
Central Tendency
Expected Paths of Inflation, Real Rate, and Central Tendency
Following One Standard Deviation Jump in Real Rate
Deviation from Long-Run Mean
Years
Inflation
Real Rate
Central Tendency
Expected Paths of Inflation, Real Rate, and Central Tendency
Following One Standard Deviation Jump in Central Tendency
Deviation from Long-Run Mean
Years
Inflation
Real Rate
Central Tendency

Figure 5

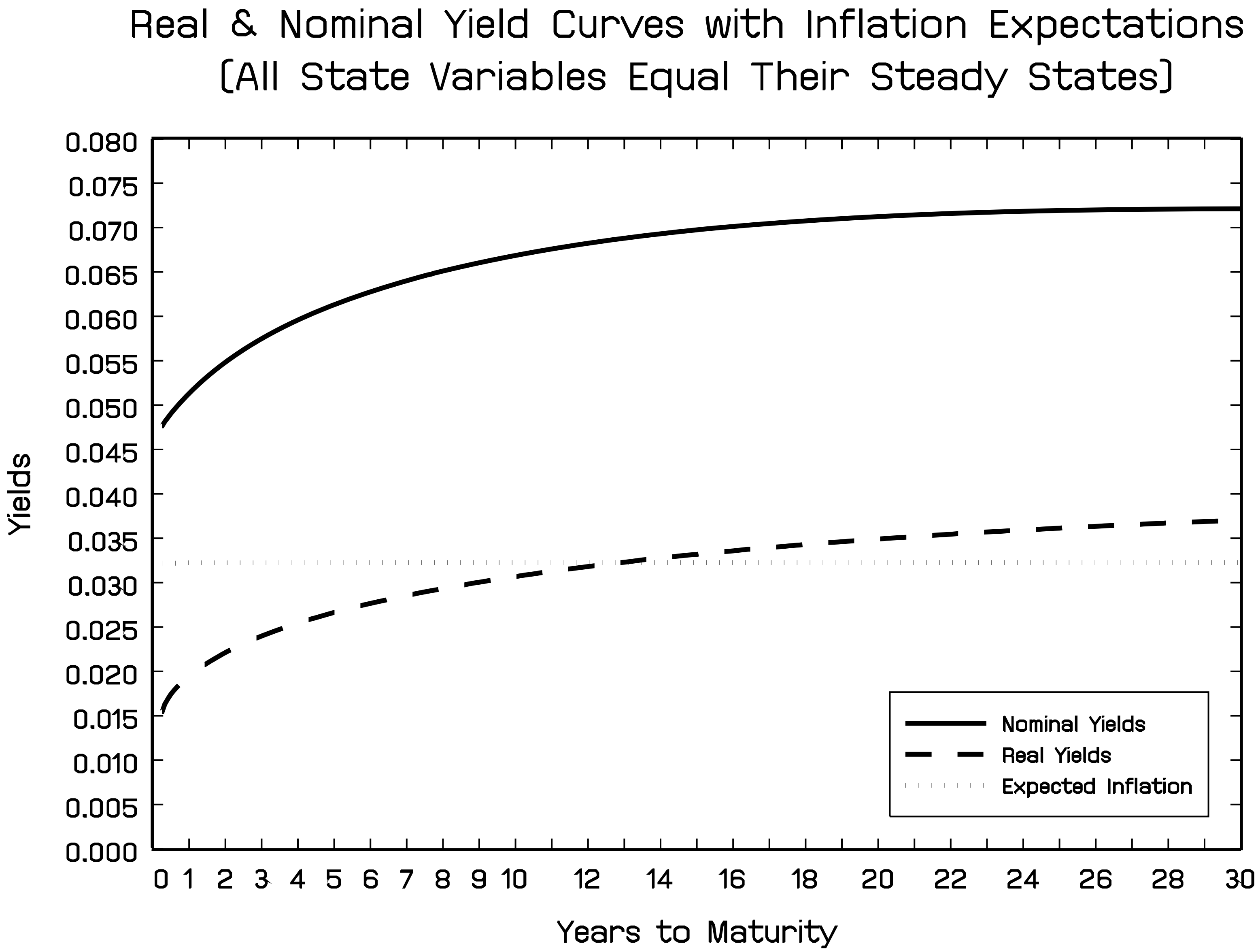
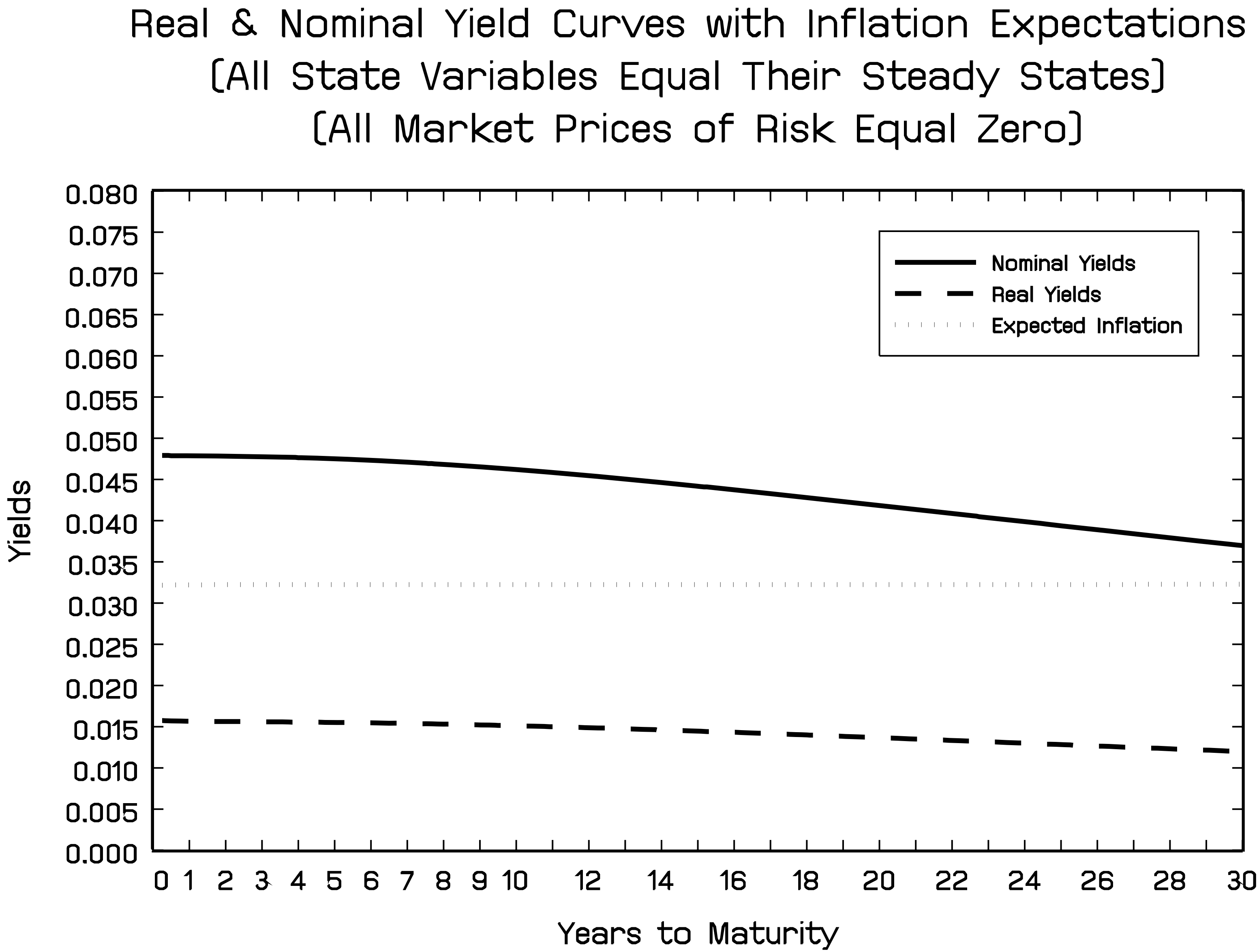
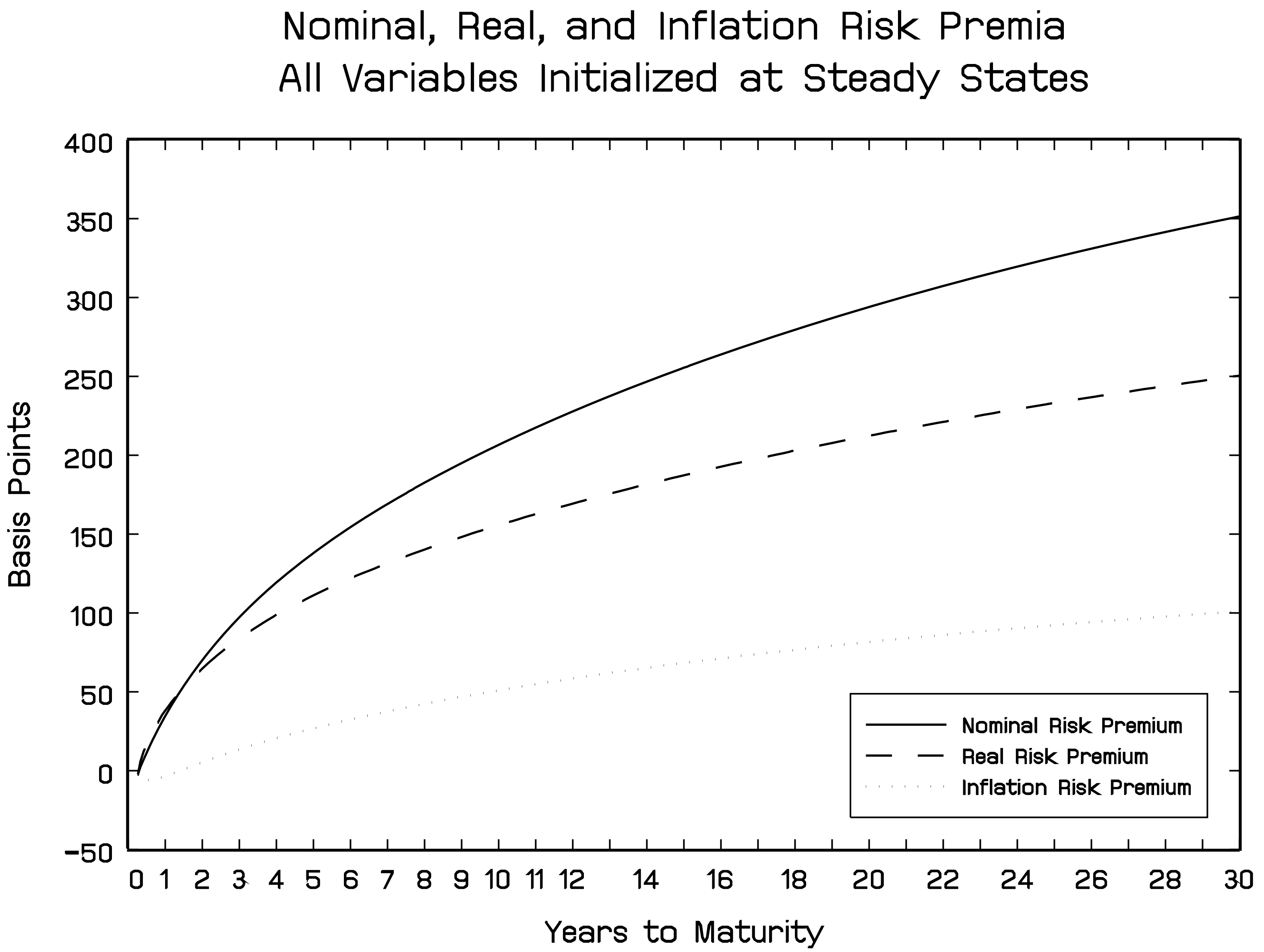

Real & Nominal Yield Curves with Inflation Expectations
(All State Variables Equal Their Steady States)
Yields
0.080
0.075
0.070
0.065
0.060
0.055
0.050
0.045
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20 22 24 26 28 30
Years to Maturity
Nominal Yields
Real Yields
Expected Inflation

Figure 6



Real & Nominal Yield Curves with Inflation Expectations
(All State Variables Equal Their Steady States)
(All Market Prices of Risk Equal Zero)
Nominal Yields
Real Yields
Expected Inflation
Yields
0.080
0.075
0.070
0.065
0.060
0.055
0.050
0.045
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20 22 24 26 28 30
Years to Maturity

Figure 7



Nominal, Real, and Inflation Risk Premia
All Variables Initialized at Steady States
Basis Points
400
350
300
250
200
150
100
50
0
-50
0
1
2
3
4
5
6
7
8
9
10
11
12
14
16
18
20
22
24
26
28
30
Years to Maturity
Nominal Risk Premium
Real Risk Premium
Inflation Risk Premium

Figure 8

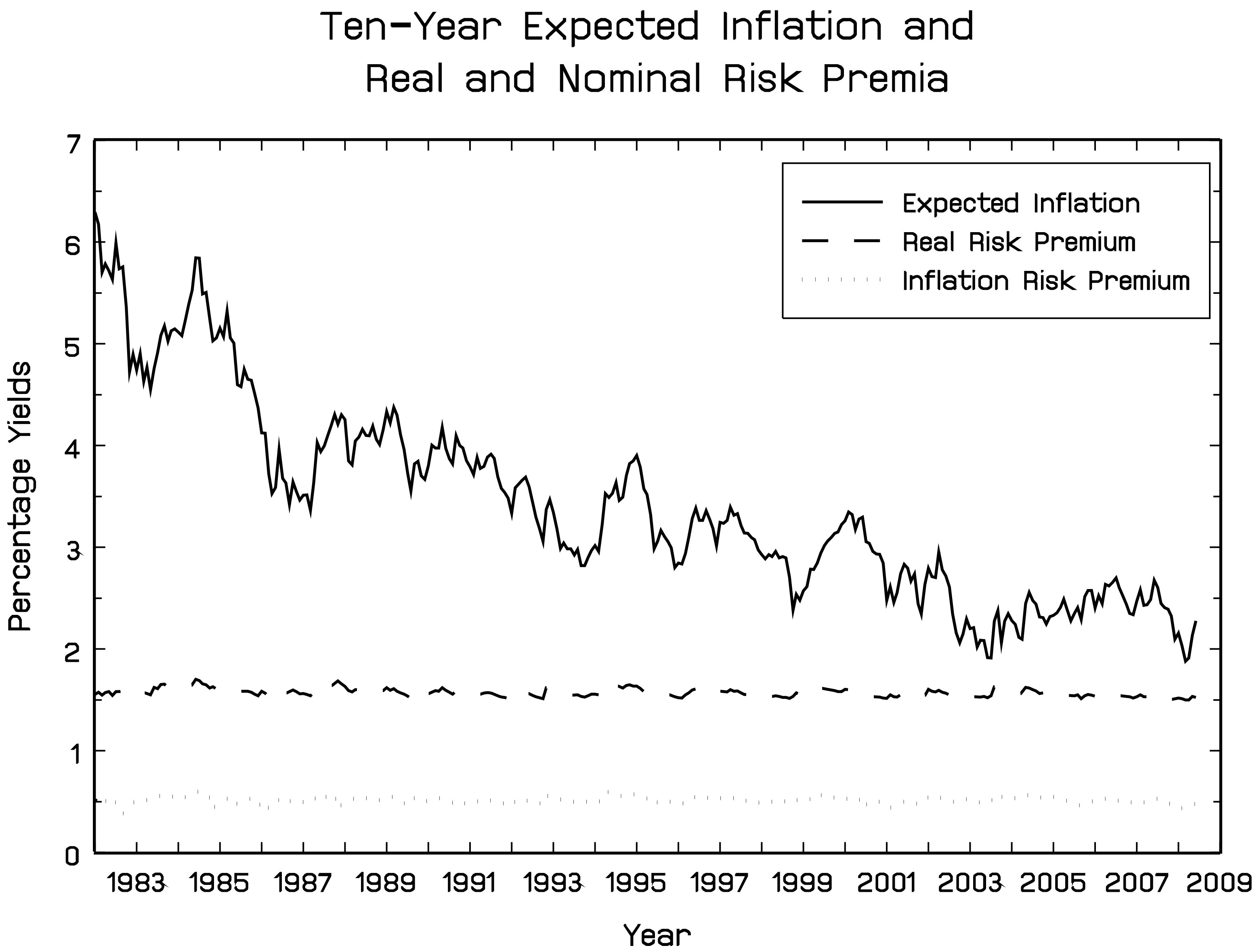

Ten-Year Expected Inflation and
Real and Nominal Risk Premia
Expected Inflation
Real Risk Premium
Inflation Risk Premium
Percentage Yields
Year
0
1
2
3
4
5
6
7
1983
1985
1987
1989
1991
1993
1995
1997
1999
2001
2003
2005
2007
2009

Figure 9

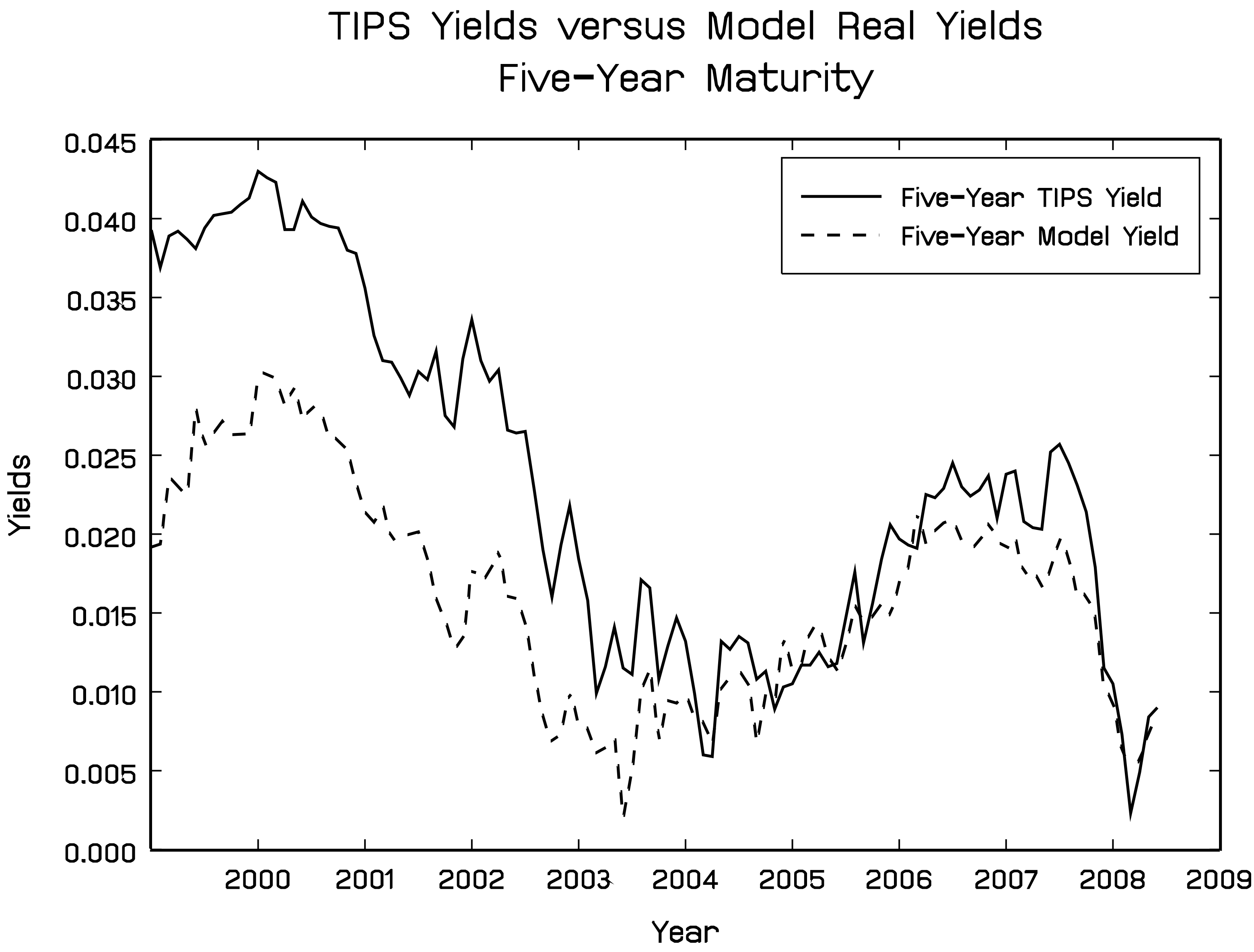
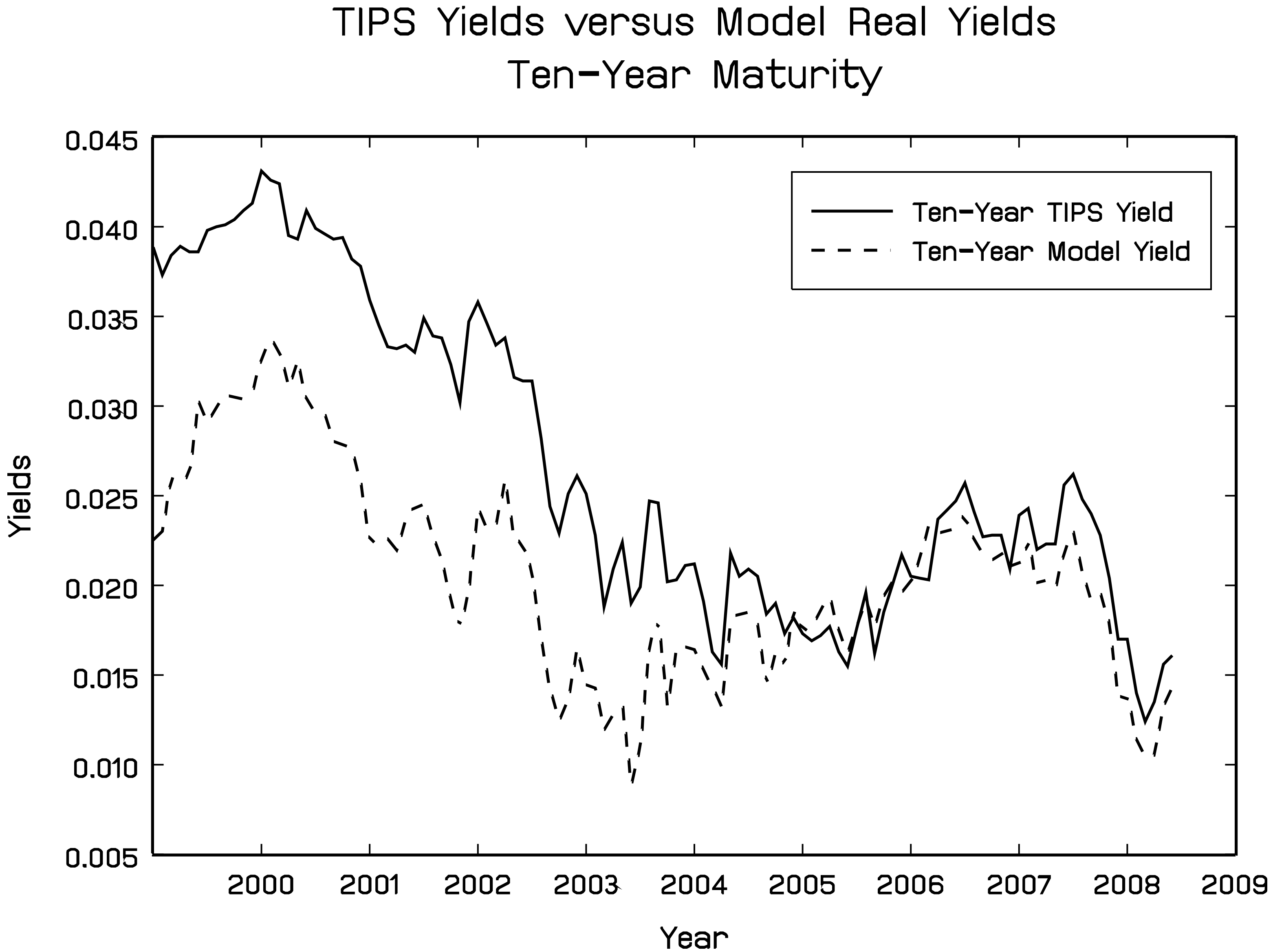

TIPS Yields versus Model Real Yields
Five-Year Maturity
Five-Year TIPS Yield
Five-Year Model Yield
Yields
0.045
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
2000
2001
2002
2003
2004
2005
2006
2007
2008
2009
Year

Figure 10



TIPS Yields versus Model Real Yields
Ten-Year Maturity
Ten-Year TIPS Yield
Ten-Year Model Yield
0.045
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
Yields
2000
2001
2002
2003
2004
2005
2006
2007
2008
2009
Year