Modal Analysis
(Structural Engineering)

Concept Statement: Every structure has a natural period. And, if an earthquake shakes at or near this period, then the structure may begin to resonate and shake wildly and fall down.

Summer Engineering Academy
Summer 2007
Engineering in Practice-University of Oklahoma

Developed by Lisa Holliday and Kat Baker
List of Materials Needed:

Exploration: Part A-Let it Shake
Each of the following is required per class:
- Shaker Table
- 5 Story Mock Structure

Concept Application
Each of the following is required per class:
- Shaker Table (per class)
- 2 Story Mock Structure (per class)
Each of the following is required per student:
- 1 calculator (per student)
- 1 computers with internet access

Activity Time Frame:
This activity will take approximately 7 hours.

Environmental Setting
This activity will be conducted at the Donald G. Fears Structural Engineering Laboratory. Computers with access to the internet will be required for students to complete the WebQuest provided.

Lesson Objectives:
- Demonstrate the effect of earthquakes on different buildings
- Use matrices to solve a problem
- Compare math results with real life results

PASS Standards:
Mathematics Process Standards:
1) Problem Solving
   2. Identify the problem from a described situation, determine the necessary data and apply appropriate problem-solving strategies.

2) Communication
   1. Use mathematical language and symbols to read and write mathematics and to converse with others.

4) Connections
   1. Link mathematical ideas to the real world.
   2. Apply mathematical problem-solving skills to other disciplines.
   3. Use mathematics to solve problems encountered in daily life.

Algebra II Content Standards:
1) Number Systems and Algebraic Operations
   3. Matrices
      a. Add, subtract, and multiply matrices to solve problems.
      b. Find the inverse and determinant of a matrix to solve problems.
      c. Use matrices to solve systems of equations.

2) Relations and Functions
   3. Solve quadratic equations by graphing, factoring, completing the square and quadratic formula.

Science Processes and Inquiry: Physics
Process Standard 1: Observe and Measure
   1. Identify qualitative and quantitative changes given conditions before, during, and after an event.

Process Standard 3: Experiment
   3. Use mathematics to show relationships within a given set of observations.

Process Standard 4: Interpret and Communicate
   1. Select appropriate predictions based on previously observed patterns of evidence.

Process Standard 5: Model
   1. Interpret a model which explains a given set of observations.
   2. Select predictions based on models.
   3. Compare a given model to the real world.

Process Standard 6: Inquiry
   1. Inquiries should lead to the formulation of explanations or models. In answering questions, students should engage in discussions and arguments that encourage the revision of their explanations, leading to further inquiry.

Vocabulary Terms
   All new terminology will be defined in the activity.

Background Knowledge
   Background knowledge will be covered in the activity.

Additional Resources and References:
http://www.okgeosurvey1.gov/Oklahoma Earthquake Information
http://www.youtube.com/watch?v=3mclp9QmCGs&mode=related&search Tacoma Narrows Bridge Video
http://www.youtube.com/watch?v=otyLaENTkHE&mode=related&search Shaker table demonstrations
http://www.youtube.com/watch?v=O2XmOXVOvo Shaker table demonstrations

Activity Procedures:
   • As a class, observe the behavior of a 5 story mock building on a shaker table
   • Calculate resonant frequencies for a 2 story mock building
   • Test calculations with mock building on shaker table
   • Use the internet to research a particular question related to earthquakes

Technology Component
   A shaker table, a multi-level mock building, a calculator, and a computer with internet access are required for this activity.

Engineering Application
   Structural Engineers use these same processes to evaluate the stability of different buildings during earthquakes so that they can design buildings that are more resistant to damage during an earthquake.

Assessment Tools
   Final calculation of resonant frequency and mode shape
   WebQuest findings
Modal Analysis
(Structural Engineering)

(Some information and pictures taken from Wikipedia.com)

INTRODUCTION

What happens to buildings during an earthquake? Well structural engineers have been studying this for years and they have discovered some interesting science and math is involved ……… yes, you heard that correctly – MATH!

Today we are going to be structural engineers!
Buildings are complicated systems and their response to an earthquake can be complicated also. Here are some examples of some different ways a tall building can move or in other words, different modes:

**Figure 1 – Computer Simulation of a Tall Building**

**Figure 2 - Swaying Back and Forth**

**Figure 3 - Swaying Side to Side**

**Figure 4 - Stretching and Compressing**

**Figure 5 - Sine Wave Shape**

**Figure 6 - Twisting**
Before we can calculate the response of complicated buildings, let’s look at a simple building. Take the following simplified 3 story building:

Here is the simplification of the building: We will explain how we are able to make this simplification later.

Here are some of the ways the building may move or deform (also known as the building’s mode shape).
**EXPLORATION**

**Part A-Let It Shake**

**Materials:**
Each of the following is required per class:
- One large shaker table
- One 5-story structure

*Visit the Fears Structural Engineering Lab on the campus of the University of Oklahoma. Attach a 5-story steel structure to the shaker table and shake the structure at various frequencies from 1 Hz to 10 Hz. What do you observe? How does the building react at different frequencies? Focus on which floors move at each frequency and record your observations in the following table. Use the example drawings provided page 3.*

<table>
<thead>
<tr>
<th></th>
<th>Comments</th>
<th>Sketches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Comments</td>
<td>Sketches</td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>----------</td>
<td></td>
</tr>
<tr>
<td>5 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 Hz</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Based on your data, what can you conclude about buildings shaken at different frequencies?

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________

Before we can understand this phenomenon, we must review some math.
Part B – Math Background

Matrices are important mathematical tools that structural engineers use to solve many complex problems including what happens to buildings during earthquakes. As you have experienced, buildings react differently when they are shaken at different frequencies. Each of these different reactions is called a Mode.

Teacher note: A modal analysis calculates the frequency modes \( (\phi_n) \) and natural frequencies of a given system. The natural frequency of a system is dependent only on the stiffness of the structure and the mass which participates with the structure (including self-weight). It is useful to know the modal frequencies of a structure as it allows you to ensure that the frequency of any applied periodic loading will not coincide with a modal frequency and hence cause resonance, which leads to large oscillations. The mode shapes describe the deflected shape of a building. Here is an example of a first and second mode of building seismic response.

Structural engineers use matrix analysis to determine the natural period of complex structures so that they can predict how a building will behave during an earthquake. Let’s investigate further matrix math!!!

Introduction to Matrix Math

Right now, take a look around you. It may not seem like math is here, but it is.

For example, let’s create a simple chart. This chart will describe how many men and women there are in the room right now, and how many of them wear glasses.

<table>
<thead>
<tr>
<th></th>
<th>Glasses</th>
<th>No Glasses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Women</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Chart 1

What if we wanted to know how many lenses the men had, and how many lenses the women had? We would take the number of men wearing glasses, multiply by two (number of lenses), and take the number of women wearing glasses and multiply by two.
What if I create a second chart which listed the number of lenses for people with glasses and people with no glasses?

<table>
<thead>
<tr>
<th></th>
<th>Number of Lenses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glasses</td>
<td>2</td>
</tr>
<tr>
<td>No Glasses</td>
<td>0</td>
</tr>
</tbody>
</table>

**Chart 2**

If I took the numbers in the first column (Labeled “Glasses”) of Chart 1 and multiplied each by two, I would have done the same thing. In mathematics, we call this type of chart, a **matrix**. When we “multiply” the charts together, we are actually doing matrix multiplication. The only difference is that we leave off the labels on the chart.

Rewrite your charts without the labels.

![Chart 1](image1)

![Chart 2](image2)

**Chart 1**

**Chart 2**

Taking a look at the men’s row first in Chart 1, we would want to multiply the number of men wearing glasses by two and the number of men not wearing glasses by 0. To do this we…

1. Take the first row, first column of Chart 1 and multiply it by the first row of Chart 2: ___
2. Take the first row, second column of Chart 1 and multiply it by the second row of Chart 2: ___
3. Add these two numbers and you have the number of lenses the men are wearing: ___
4. Put your answer in the Results Chart below.

Now we’ll repeat this with the women’s row.

1. Take the second row first column of Chart 1 and multiply it by the first row of Chart 2. ___
2. Take the second row second column of Chart 1 and multiply it by the second row Chart 2. ___
3. Add these two numbers and you have the number of lenses the women are wearing. ___
4. Put your answer in the Results Chart below.

Our result will be a matrix (chart) of the form

\[
\begin{bmatrix}
\text{Men's Lenses} & \text{Women's Lenses}
\end{bmatrix}
\]

**Results Chart**
Putting this all together, our equation is (first matrix) times (second matrix) = result matrix

\[
\begin{bmatrix}
\square & \square & \\
\square & \square & \\
\end{bmatrix}
\begin{bmatrix}
\square \\
\square \\
\end{bmatrix}
= 
\begin{bmatrix}
\square \\
\square \\
\end{bmatrix}
\]

Next we’re going to see why matrix multiplication is so useful and important.

Before we work an example, reflect on your observations of our multi-level mock structure on the shaker table. What are the two structural components of that structure? Lead students into saying that the floors and the columns make up the structure. Note: For this type of matrix analysis, the two most important features are a building’s mass and its stiffness. We will discuss this in greater detail in the next section.
CONCEPT DEVELOPMENT

In the previous section we discussed two important structural components of the multilevel mock structure. These components were __floors____ and ___columns___. Teacher note: For simplicity, we will replace “floors” with “mass”, and we replace “columns” with “stiffness.” Let’s take a look at some of the simplifications and assumptions engineers make when performing modal analysis. For this type of analysis the two most important features are a building’s mass and stiffness.

Mass

To further simplify our problem, we assume that all the mass of a building is clumped into one spot on each of the floors and the roof (as indicated by the red dots in the Exploration); this makes the calculations more simple. To clarify, we will label our building with levels instead of floors and ceilings. For example, here is a two story building:

If we assume the mass of the building is located at level 1 and level 2. Then a sketch of the buildings mass would look like this:

Notice there is no mass for the ground floor (Level 0) because it sits directly on the ground.
Let's say level 2 has \( \frac{1}{2} \) the mass of level 1. Our Sketch would then look like this:

![Sketch showing levels 1 and 2 with masses m_1 = 1m and m_2 = \( \frac{1}{2} \)m]

The chart form of the mass matrix for this situation would look like this:

<table>
<thead>
<tr>
<th></th>
<th>Caused by level 1</th>
<th>Caused by level 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1 mass</td>
<td>( m_1 )</td>
<td>0</td>
</tr>
<tr>
<td>Level 2 mass</td>
<td>0</td>
<td>( m_2 )</td>
</tr>
</tbody>
</table>

The \( m_1 \) position describes the mass of level 1 caused by level 1. The \( m_2 \) position describes the mass of level 2 caused by level 2. The other two values are zero because the roof won't affect the floor, and the floor won't affect the roof.

The mathematical notation of the mass matrix for this situation would look like this:

\[
[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}
\]

So our mass matrix becomes:

\[
[M] = \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}
\]
Stiffness

For purposes of determining the stiffness of the building, a sketch of the columns would look like this:

We have sketched and labeled two important components of a building, these are mass and stiffness. But, what exactly are mass and stiffness?

**Mass** (m) - In informal everyday usage, mass is more commonly referred to as weight, but in physics and engineering weight strictly means the size of the gravitational pull on the object; that is, how heavy it is, measured in units of force. Mass is independent of gravity. For example, a bowling ball would have a different weight on the moon, because the gravitation pull is different, but it would still have the same mass on the moon. For our example, we will focus on the mass of each floor of our mock structure.

**Force** (F) – A force is influence that may cause a body to accelerate. It may be experienced as a lift, a push, or a pull. A force may also cause rotation or deformation of the body. Earthquakes apply loads or forces to buildings. An earthquake shakes the ground on which a building sits. This shaking or movement has a displacement, velocity, and acceleration. From physics we know that: force = mass x acceleration or F = ma.

In the case of a building, the mass is related to the weight of the building and the acceleration comes from the earthquake and these two combine to create loads or forces on the building. Force is directly related to deflection (or deformation of the buildings) and this can cause a building to fail or break if it exceeds the strength of the building. For our example, we will focus on the force that results from the shaker table shaking our mock structure.
Deflection (δ) - In engineering mechanics, deflection is a term that is used to describe the degree to which a structural element is displaced under a load. The deflection of a member under a load is directly related to the slope of the deflected shape of the member under that load and can be calculated by integrating the function that mathematically describes the slope of the member under that load. For our example, we will focus on the movement or deflection of our mock structure.

![Deflection Diagram]

Stiffness (k) - Stiffness is the resistance of an elastic body to deflection or deformation by an applied force. It is an extensive material property. The stiffness k of a body that deflects a distance δ under an applied force F is

\[ k = \frac{P}{\delta} \]

Stiffness is very similar to modulus of elasticity. In our case, k defines the stiffness of the structure as a whole and includes the contributions of all the materials in the structure. For our example, we will focus on the stiffness of the metal columns of our mock structure.
Young's Modulus (or modulus of elasticity E) – In solid mechanics, Young’s modulus (E) is the measure of the stiffness of a given material. It is also known as modulus of elasticity or tensile modulus. It is defined as the ratio of the change in stress over the change in strain. It is most important to realize it defines the stiffness of a material or in other words, how hard a material is to deform. The modulus of elasticity for some common materials is listed in the following table. The higher the value the more force it takes to deform a material or the stiffer the material. For our example, we will focus on the type of material of the columns in our mock structure; this material is steel.

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s modulus (E) in lb/in² (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rubber</td>
<td>1,500-15,000</td>
</tr>
<tr>
<td>Nylon</td>
<td>150,000-435,000</td>
</tr>
<tr>
<td>Oak wood (along the grain)</td>
<td>1,600,000</td>
</tr>
<tr>
<td>High strength concrete (in compression)</td>
<td>6,500,000</td>
</tr>
<tr>
<td>Glass</td>
<td>10,400,000</td>
</tr>
<tr>
<td>Steel</td>
<td>30,000,000</td>
</tr>
<tr>
<td>Diamond</td>
<td>1500,000,000</td>
</tr>
</tbody>
</table>
Now after all of our simplifications for mass and stiffness, our simplified sketch of the building would look like this:

![Building Sketch](image)

The lines on the bottom specify that the bottom of the building is attached to the ground.

Now we have to determine our stiffness matrix.

![Stiffness Matrix](image)

$k$ is the generic stiffness value, $k = \frac{12(EI)}{H^3}$

$k_1$ is the stiffness of all the elements that attach to $m_1$.

So considering the columns for both sides, $k_1 = (2) \frac{12(EI)}{H^3}$ or $k_1 = \frac{24(EI)}{H^3}$

and considering the columns for both sides, $k_2 = (1) \frac{12(EI)}{H^3}$ or $k_2 = \frac{12(EI)}{H^3}$

In our mass matrix, the mass of one level does not affect the mass of another level. This is **not** the case with stiffness. The stiffness matrix for a building is defined as:

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$
In the preceding equations, we know that $k$ represents the stiffness and $E$ represents the Modulus of Elasticity of a column. But, what does $I$ represent? $I$ symbolizes the column’s moment of inertia, and it indicates the geometry of the column. Continue reading the next paragraph.

**Moment of Inertia ($I$)** - The moment of inertia of an object about a given axis describes how difficult it is to change its angular motion about that axis. For example, consider two discs of the same mass, one large and one small in radius. Assuming that there is uniform thickness and mass distribution, the larger radius disc requires more effort to accelerate it (i.e. change its angular motion) because its mass is effectively distributed further from its axis of rotation. Conversely, the smaller radius disc takes less effort to accelerate it because its mass is distributed closer to its axis of rotation. Quantitatively, the larger disc has a larger moment of inertia, whereas the smaller disc has a smaller moment of inertia.

So in short, moment of inertia is an object’s resistance to rotation or bending. For our example, we will focus on the shape of the columns of our mock structure to determine the mock structure’s moment of inertia.

$I$ (moment of inertia) for a rectangular shape resisting rotation about its neutral axis (N.A.) (the neutral axis about which bending is occurring) is described in terms of the base dimension ($b$) and the height dimension ($h$):

\[
I = \frac{bh^3}{12}
\]

This can be demonstrated by comparing the strength of a 2” x 4” upright and on its side.

Record your observations by comparing the apparent strength of a 2” x 4” upright and on its side.

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____________________________________________________________________________
Equation of Motion

Up to this point we have only talked about stationary buildings, now let’s look at moving buildings. From equations of motion, we can derive a formula that relates stiffness, mass, natural frequency, and mode shapes.

\[ K \phi_n = \omega_n^2 M \phi_n \]

Where \( \omega_n \) is the angular natural frequency. Angular frequency (\( \omega \)) is the measure of how fast an item is spinning.

In our demonstration and example problem, nothing is spinning, but angular frequency is related to period (T) and frequency (f) as follows: \( \omega = \frac{2\pi}{T} = 2\pi f \)

If we were to take the way this circular motion changes and plot it on a straight line, then it is harmonic. The way buildings moves back and forth can be thought of as harmonic; and therefore, we have to explore some terms that describe harmonic motions such as period (T) and frequency (f).

The period (T) is the amount of time it takes to complete one cycle.
In *physics*, resonance is the tendency of a system to oscillate at maximum amplitude at a certain frequency. This frequency is known as the system's *resonant frequency*. When damping is small, the resonant frequency is approximately equal to the natural frequency ($\omega_n$) of the system, which is the frequency of free vibrations.

In matrix form this equation would be:

$$[K] \varphi_n = \omega_n^2 [M] \varphi_n$$

We can use this formula to solve for the mode shapes ($\varphi_n$) and natural frequencies ($\omega_n$).

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**CONCLUSION**

1. From your observations in the Exploration, how do buildings respond to different frequencies? *Buildings respond differently to different frequencies (earthquakes).*

2. From your observations and discussions, what are two major components of a structure that determines the structure’s response to an earthquake? *Mass and stiffness*

3. What mathematical tool can be used to determine how a building will respond during an earthquake? *A matrix.* Briefly provide a description of this mathematical tool. *A matrix is an organizational tool that allows one to predict a building’s response to an earthquake.*
CONCEPT APPLICATION

Example:
A two story building is shown below. The mass of the building can be idealized as two lumps, one at the first level and one on the second level. The second level only weighs half as much as the first level. The earthquake shaking force is shown as P(t). The columns have height (H), modulus of elasticity (E), and moment of inertia (I).

\[ \begin{bmatrix} P_2(t) \\ m/2 \end{bmatrix} \quad \begin{array}{c} \bullet \\ \hline \hline \bullet \end{array} \quad \begin{bmatrix} P_1(t) \\ m \end{bmatrix} \]

The mass matrix for this structure is:
\[ [M]= \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \]

\[ m=0.0095 \text{ lb-s}^2/\text{in} \]

\[ b=1\text{in} \]
\[ h=1/16\text{ in} \]
\[ H=12\text{in} \]

\[ I = 2 \frac{bh^2}{12} = 2 \frac{(1\text{in})(1/16\text{in})^3}{12} = 0.00004069\text{in}^4 \]

The stiffness at level 1 is:
\[ k_1 = \frac{24(EI)}{h^3} \]

The stiffness at level 2 is:
\[ k_2 = \frac{12(EI)}{h^3} \]
Putting these two values into the global stiffness matrix:

\[
[K] = \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\]

In matrix form the equation of motion is:

\[
[K] \varphi_n = \omega_n^2 [M] \varphi_n
\]

Typically in algebra we could cancel the two \( \varphi_n \), but when dealing with matrices they cannot cancel out. In order to solve this problem, we must first move everything to the left side of the equation.

\[
[K] - \omega_n^2 M \varphi_n = 0
\]

\( \varphi_n \) are not equal to zero. If they were equal to zero, then nothing would be happening with the building and what fun would that be?

So \( [K] - \omega_n^2 M \) must equal zero. Because \( K \) and \( M \) are matrices we can’t just set \( [K] - \omega_n^2 M = 0 \).

Notice that \( [K] - \omega_n^2 M \) is also a matrix. What we do is set the determinant of this to equal zero.

\[
\text{det} [K] - \omega_n^2 M = 0
\]

The determinant is just a mathematical device we use to solve these types of equations. The determinant of a matrix is...

\[
\text{det} \begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix} = AD - BC
\]

Once we know the \( K \) and \( M \) matrices, we can plug them in and solve for \( (\omega_n) \).

\( \omega_n \) will help us determine at what frequencies the building will shake and \( \varphi_n \) will show the shape.

We can use this formula to solve for the mode shapes \( (\varphi_n) \) and natural frequencies following a series of steps as shown.

1. Find \( [K] - \omega_n^2 M \)

\[
\begin{bmatrix}
  \frac{36(El)}{H^3} & -\frac{12(El)}{H^3} \\
  -\frac{12(El)}{H^3} & \frac{12(El)}{H^3}
\end{bmatrix} - \begin{bmatrix}
  m\omega_n^2 & 0 \\
  0 & \frac{1}{2}m\omega_n^2
\end{bmatrix} = \begin{bmatrix}
  \frac{36(El)}{H^3} - m\omega_n^2 & -\frac{12(El)}{H^3} \\
  -\frac{12(El)}{H^3} & \frac{12(El)}{H^3} - \frac{1}{2}m\omega_n^2
\end{bmatrix}
\]
2. Substitute \( \alpha = \frac{12(El)}{IR^2} \) and \( \beta = m\omega_n^2 \)

\[
\begin{bmatrix}
3\alpha - \beta & -\alpha \\
-\alpha & \alpha - \frac{1}{2} \beta
\end{bmatrix}
\]

3. Take the determinate and set it equal to 0:

\[
\det \begin{bmatrix}
3\alpha - \beta & -\alpha \\
-\alpha & \alpha - \frac{1}{2} \beta
\end{bmatrix} = 0
\]

\[
(3\alpha - \beta)(\alpha - \frac{1}{2} \beta) - \alpha^2 = 0
\]

4. Multiply this out to get:

\[
\frac{1}{2} \beta^2 - \frac{5}{2} \alpha \beta + 2\alpha^2 = 0
\]

Multiply this equation by two to make it simpler.

\( \beta^2 - 5\alpha \beta + 4\alpha^2 = 0 \)

We can solve this using the quadratic equation

\[
\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

In our case \( a = 1, b = -5\alpha, c = 4\alpha^2 \)

\[
\beta = \frac{5\alpha \pm \sqrt{25\alpha^2 - 16\alpha^2}}{2} = \frac{5\alpha \pm \sqrt{9\alpha^2}}{2} = \frac{(5 \pm 3)\alpha}{2}
\]

\( \beta = 4\alpha \) and \( \beta = 1\alpha \)

5. Look at the first solution \( \beta = 4\alpha \)

Plug in \( \alpha = \frac{12(El)}{IR^2} \) and \( \beta = m\omega_n^2 \)

\[
m\omega_n^2 = 4 \frac{12(El)}{IR^2} = 48 \frac{(El)}{IR^2}
\]

6. Solve for \( \omega_{n1} = \sqrt{48 \frac{(El)}{mIR^2} - \frac{48(0.00000018 \text{ lb-in})}{(0.0038^2 \text{ in}^2)(12 \text{ in})^2}} = 59.7 \text{ rad/s} \)

7. Look at the second solution \( \beta = 1\alpha \)

Plug in \( \alpha = \frac{12(El)}{IR^2} \) and \( \beta = m\omega_n^2 \)

\[
m\omega_n^2 = 12 \frac{(El)}{IR^2}
\]
8. Solve for $\omega_{n2} = \sqrt{\frac{12(El)}{mH^3}} = 3.750 \sqrt{\frac{(El)}{mH^3}} = \sqrt{\frac{12(30,000,000 \text{ lb})(0.0004 \text{ in}^4)}{(0.0930 \text{ in})(12 \text{ in})^3}} = 29.9 \text{ rad/s}$

Now that we have the natural circular frequencies ($\omega_n$), we can solve for the natural frequencies ($f_n$) and the natural periods ($T_n$).

<table>
<thead>
<tr>
<th></th>
<th>$\omega_{n1} = 59.7 \text{ rad/s}$</th>
<th>$\omega_{n2} = 29.9 \text{ rad/s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>g. $f_n = \frac{\omega_n}{2\pi \text{ rad/s}}$</td>
<td>$9.5 \text{ Hz}$</td>
<td>$4.8 \text{ Hz}$</td>
</tr>
<tr>
<td>10. $T_n = \frac{1}{f_n}$</td>
<td>$0.11 \text{ s}$</td>
<td>$0.21 \text{ s}$</td>
</tr>
</tbody>
</table>

11. Each of the natural frequencies will have a particular mode shape associated with them.

For $\omega_{n1}$ we have $\phi_{n1} = \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix}$, for $\omega_{n2}$ we have $\phi_{n2} = \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix}$.

12. To find the mode shapes ($\phi_n$), individually substitute the $\omega_n$ into the equation $[K - \omega_n^2 M][\phi_n] = 0$.

Substitute $\omega_{n1}$ first.

$$\begin{bmatrix} \frac{36(El)}{H^3} - m\omega_{n1}^2 \\ -\frac{12(El)}{H^3} \end{bmatrix} \begin{bmatrix} -\frac{12(El)}{H^3} \\ \frac{1}{2} m\omega_{n1}^2 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0$$

Remember that $m\omega_{n1}^2 = \frac{48 (El)}{H^3}$.

$$\begin{bmatrix} \frac{36(El)}{H^3} - \frac{48(El)}{H^3} \\ -\frac{12(El)}{H^3} \end{bmatrix} \begin{bmatrix} -\frac{12(El)}{H^3} \\ \frac{1}{2} \frac{48(El)}{H^3} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} -\frac{12(El)}{H^3} \\ -\frac{12(El)}{H^3} \end{bmatrix} \begin{bmatrix} -\frac{12(El)}{H^3} \\ \frac{24(El)}{H^3} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = \begin{bmatrix} -\frac{12(El)}{H^3} \\ -\frac{12(El)}{H^3} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix}$$

The $-\frac{12(El)}{H^3}$ can be factored out since it is in each part of the matrix.

$$-\frac{12(El)}{H^3} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0$$
13. $\frac{12EI}{H^3}$ is just a number and so we can divide both sides by it. Notice that $m$, $E$, $I$, and $H$ have cancelled out. What matters is the relationship between the masses (here $\frac{1}{2}$) and the K matrix.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0$$

14. Use the matrix multiplication you reviewed earlier to multiply the top row.

$$\phi_{11} + \phi_{12} = 0$$
$$\phi_{12} = -\phi_{11}$$

If $\phi_{11} = 1$ then $\phi_{12} = -1$

The bottom row would give you the same results so multiplying it out is not necessary.

15. This gives us $\phi_{n1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which looks like

16. Substitute $\omega_{n2}$ next.

$$\begin{bmatrix} \frac{36EI}{H^3} - m\omega_{n2}^2 & -\frac{12EI}{H^3} \\ -\frac{12EI}{H^3} & \frac{H^3}{12EI} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$

Remember that $m\omega_{n2}^2 = \frac{12EI}{H^3}$

$$\begin{bmatrix} \frac{36EI}{H^3} - \frac{12EI}{H^3} & -\frac{12EI}{H^3} \\ -\frac{12EI}{H^3} & \frac{12EI}{H^3} + \frac{H^3}{12EI} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{24EI}{H^3} & -\frac{12EI}{H^3} \\ -\frac{12EI}{H^3} & \frac{H^3}{12EI} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \frac{24EI}{H^3} & -\frac{12EI}{H^3} \\ -\frac{12EI}{H^3} & \frac{6EI}{H^3} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$

The $\frac{12EI}{H^3}$ can be factored out since it is in each part of the matrix.

$$\frac{12EI}{H^3} \begin{bmatrix} 2 & -1 \\ -1 & 0.5 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$
17. \( \frac{12(2\pi)}{H^2} \) is just a number and so we can divide both sides by it.

\[
\begin{bmatrix}
2 & -1 \\
-1 & 0.5
\end{bmatrix}
\begin{bmatrix}
\phi_{21} \\
\phi_{22}
\end{bmatrix} = 0
\]

18. Use the matrix multiplication you reviewed earlier to multiply the top row.

\( 2\phi_{21} - \phi_{22} = 0 \)

\( 2\phi_{21} = \phi_{22} \)

If \( \phi_{21} = 1 \) then \( \phi_{22} = 2 \)

Again, the bottom row would give you the same results so it’s not necessary.

19. This gives us \( \phi_{n2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) which looks like

![Diagram](image)
APPLICATION PROBLEM

Given the following structure (with the same K-matrix as our earlier example):

\[ M_1 = 0.0095 \frac{lb \cdot s^2}{\text{in}} \]
\[ M_2 = 0.0095 \frac{lb \cdot s^2}{\text{in}} \]

b=1 in
h=1/16 in
H=12 in

Look up (in this activity) any other quantities that you need.
The structure is made out of steel.

When you are finished, build a model to these specs and shake the model at the natural frequencies and see what happens.

1. Find \([K - \omega_n^2 M]\)

\[
\begin{bmatrix}
\frac{36EI}{H^3} & -\frac{12EI}{H^3} \\
\frac{12EI}{H^3} & \frac{12EI}{H^3}
\end{bmatrix}
\begin{bmatrix}
m \omega_n^2 & 0 \\
0 & m \omega_n^2
\end{bmatrix}
= \begin{bmatrix}
\frac{36EI}{H^3} - m \omega_n^2 & \frac{-12EI}{H^3} \\
\frac{-12EI}{H^3} & \frac{-12EI}{H^3} - m \omega_n^2
\end{bmatrix}
\]

2. Substitute \(\alpha = \frac{12EI}{H^3}\) and \(\beta = m \omega_n^2\)

\[
\begin{bmatrix}
3\alpha - \beta & -\alpha \\
-\alpha & \alpha - \beta
\end{bmatrix}
\]
3. Take the determinate and set it equal to 0:

\[(3\alpha - \beta)(\alpha - \beta) - \alpha^2 = 0\]

4. Multiply this out to get:

\[\beta^2 - 4\alpha\beta + 2\alpha^2 = 0\]

We can solve this using the quadratic equation

\[\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

In our case \(a = \), \(b = \), \(c = \)

\[a=1, \ b=-4\alpha, \ c=2\alpha^2\]

\[\beta = \frac{(-4\alpha) \pm \sqrt{(-4\alpha)^2 - 4(1)(2\alpha^2)}}{2(1)} = \frac{4\alpha \pm \sqrt{16\alpha^2 - 8\alpha^2}}{2} = \frac{4\alpha \pm \sqrt{8\alpha^2}}{2} = (2 \pm \sqrt{2})\alpha\]

\[\beta = 3.414 \text{ and } \beta = 0.586\alpha\]

5. Look at the first solution \(\beta = \)

\[\beta = 3.414\alpha\]

Plug in \(\alpha = \frac{12(\beta)}{H^3}\) and \(\beta = m\omega_n^2\)

\[m\omega_n^2 = 3.414 \frac{12(\beta)}{H^3}\]

\[\omega_{n1}^2 = 3.414 \frac{12(\beta)}{mH^3}\]
6. Solve for $\omega_{n1} =$

$$\omega_{n1} = \sqrt{\frac{3.414}{mH^2}} \sqrt{\frac{12EI}{\frac{12}{4}(\frac{L}{2})^2(12\text{ in})^3}} = 55.2 \frac{\text{rad}}{s}$$

7. Look at the second solution $\beta =$

$$\beta = 0.586\alpha$$

Plug in $\alpha = \frac{12EI}{H^3}$ and $\beta = m\omega_n^2$

$$m\omega_{n2}^2 = 0.586 \frac{12EI}{H^3}$$

$$\omega_{n2}^2 = 0.586 \frac{12EI}{mH^3}$$

8. Solve for $\omega_{n2} =$

$$\omega_{n2} = \sqrt{0.586 \frac{12EI}{mH^3}} = \sqrt{0.586 \frac{12 \left( \frac{30,000,000}{\text{in}^2} \left( \frac{0.0004069}{\text{in}^4} \right) \right)}{(0.095 \frac{\text{lb} \cdot \text{s}^2}{\text{in}})(12 \text{ in})^3}} = 22.9 \frac{\text{rad}}{s}$$

Now that we have the natural circular frequencies ($\omega_n$), we can solve for the natural frequencies ($f_n$) and the natural periods ($T_n$).

<table>
<thead>
<tr>
<th></th>
<th>$\omega_{n1} = 55.2 \text{ rad/s}$</th>
<th>$\omega_{n2} = 22.9 \text{ rad/s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. $f_n = \frac{\omega_n}{2\pi \text{ rad}}$</td>
<td>$8.8 \text{ 1/s} = 8.8 \text{ Hz}$</td>
<td>$3.6 \text{ 1/s} = 3.6 \text{ Hz}$</td>
</tr>
<tr>
<td>10. $T_n = \frac{1}{f_n}$</td>
<td>$.11 \text{ s}$</td>
<td>$.27 \text{ s}$</td>
</tr>
</tbody>
</table>
11. Each of the natural frequencies will have a particular mode shape associated with them.

For \( \omega_{n1} \) we have \( \phi_{n1} = \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} \) for \( \omega_{n2} \) we have \( \phi_{n2} = \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} \).

12. To find the mode shapes \( (\phi_{n}) \) individually substitute the \( \omega_{n} \) into the equation \( [K - \omega_{n}^2 M] \phi_{n} = 0 \).

Substitute \( \omega_{n1} \) first.

\[
\begin{bmatrix}
\frac{36(EL)}{H^3} - m\omega_{n1}^2 & -\frac{12(EL)}{H^3} \\
-\frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} - m\omega_{n1}^2
\end{bmatrix}
\begin{bmatrix}
\phi_{11} \\
\phi_{12}
\end{bmatrix}
= 0
\]

Note: \( m\omega_{n1}^2 = 3.414 \frac{12(EL)}{H^2} \)

\[
\begin{bmatrix}
\frac{36(EL)}{H^3} - 3.414 \frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} \\
-\frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} - 3.414 \frac{12(EL)}{H^3}
\end{bmatrix}
\begin{bmatrix}
\phi_{11} \\
\phi_{12}
\end{bmatrix}
= 0
\]

13. Factor out \( \frac{12(EL)}{H^3} \) since every element in the matrix has it.

\[
\frac{12(EL)}{H^3} \begin{bmatrix}
3 - 3.414 & -1 \\
-1 & 1 - 3.414
\end{bmatrix}
\begin{bmatrix}
\phi_{11} \\
\phi_{12}
\end{bmatrix}
= 0
\]

\[
\frac{12(EL)}{H^3} \begin{bmatrix}
-0.414 & -1 \\
-1 & -2.414
\end{bmatrix}
\begin{bmatrix}
\phi_{11} \\
\phi_{12}
\end{bmatrix}
= 0
\]

14. \( \frac{12(EL)}{H^3} \) is just a number and so we can divide both sides by it. Notice that \( m, E, I, \) and \( H \) have cancelled out.

What matters is the relationship between the masses (here 1) and the \( K \) matrix.

\[
\begin{bmatrix}
-0.414 & -1 \\
-1 & -2.414
\end{bmatrix}
\begin{bmatrix}
\phi_{11} \\
\phi_{12}
\end{bmatrix}
= 0
\]

15. Use the matrix multiplication you reviewed earlier to multiply the top row.

\[
-0.414 \phi_{11} - \phi_{22} = 0
\]

\[
\phi_{22} = -0.414 \phi_{11}
\]
If $\phi_{11} = 1$ then $\phi_{12} =$

$$\phi_{12} = -0.414$$

The bottom row would give you the same results so multiplying it out is not necessary.

16. This gives us $\phi_{n1} = \begin{bmatrix} 1 \\ -0.414 \end{bmatrix}$ which looks like

This is what the building would look like if shaken at a frequency of 8.8 Hz

17. Substitute $\omega_{n2}$ next.

$$\begin{bmatrix} \frac{36(EL)}{H^3} - m\omega_{n2}^2 & -\frac{12(EL)}{H^3} \\ -\frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} - m\omega_{n2}^2 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$

Note: $m\omega_{n2}^2 = 0.586 \frac{12(EL)}{H^3}$

$$\begin{bmatrix} \frac{36(EL)}{H^3} - 0.586 \frac{12(EL)}{H^3} & -\frac{12(EL)}{H^3} \\ -\frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} - 0.586 \frac{12(EL)}{H^3} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$

Factor out $\frac{12(EL)}{H^3}$.

$$\frac{12(EL)}{H^3} \begin{bmatrix} 3 - 0.586 & -1 \\ -1 & 1 - 0.586 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$

$$\frac{12(EL)}{H^3} \begin{bmatrix} 2.414 & -1 \\ -1 & 0.414 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$
18. \(\frac{12EI}{H^2}\) is just a number and so we can divide both sides by it.

\[
\begin{bmatrix}
2.414 & -1 \\
-1 & 0.4141
\end{bmatrix}
\begin{bmatrix}
\phi_{21} \\
\phi_{22}
\end{bmatrix} = 0
\]

19. Use the matrix multiplication you reviewed earlier to multiply the top row.

\[
\begin{align*}
2.414 - \phi_{22} &= 0 \\
\phi_{22} &= 2.414 \phi_{21}
\end{align*}
\]

If \(\phi_{21} = 1\) then \(\phi_{22} =

\[
\phi_{22} = 2.414
\]

Again, the bottom row would give you the same results so it’s not necessary.

20. This gives us \(\phi_{n2} = \begin{bmatrix} 1 \\ 2.414 \end{bmatrix}\) which looks like

\[
\phi_{n2} = \begin{bmatrix} 1 \\ 2.414 \end{bmatrix}
\]

This is what the building would look like if shaken at a frequency of 3.6 Hz

What did you learn about a building’s response to an earthquake? Responses will vary.
WEBQUEST

Advanced Earthquake Topics

Introduction
Earthquake engineering is a broad field and we have only covered one small element. We haven’t even talked about what makes earthquakes happen, how earthquakes differ, infamous earthquakes, or how we measure the size of an earthquake. Let’s take some time, and learn a little more about earthquakes.

The Task
Pick one of the following questions to research online. Your response should be as complete as possible and include the references you used.

1. What is the Richter scale and how does it relate to earthquakes?

2. What is resonance? Can you find an example of a structural failure by resonance during an earthquake?

3. What is the significance of the Old Tacoma Narrows bridge failure?

4. Describe the following earthquakes:
   1985 Mexico City
   1994 North Ridge California
   1812 New Madrid, Missouri

5. Where and when was the largest recorded earthquake? The deadliest? The most costly ($)?

6. What is meant by “the ring of fire?”

7. What is the relationship between volcanoes and earthquakes?

8. How does soil type affect the results of earthquakes?
The Process
1. Pick one of the eight questions to answer and explore. Get teacher approval before continuing.
2. Research your question on-line. Try searching for the answers using the following: Google, AltaVista, Wikipedia, or the USGS web-site.
3. After you have researched and found an answer or answers to your question, write what you have learned and provide a list of references used.
4. When everyone is finished, we will share our answers.

Evaluation

Use the following table to determine the level of knowledge you gained:

<table>
<thead>
<tr>
<th>Stated Objective (your question)</th>
<th>Beginning 1</th>
<th>Developing 2</th>
<th>Accomplished 3</th>
<th>Exemplary 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A general/short answer to the question</td>
<td>A general/short answer to the question and a few other small facts.</td>
<td>A complete answer to the question.</td>
<td>A complete answer to the question and extensive background information and possibly purposing a new questions.</td>
</tr>
</tbody>
</table>

Conclusion

By answering one of the assigned questions, you have explored one more aspect of earthquake engineering. You can apply this same process to any field or topic you wish to explore. Simply think of a question you would like to know the answer and then find the answer, or even better find the answer and a new question!
Modal Analysis  
(Structural Engineering) 

(Some information and pictures taken from Wikipedia.com) 

INTRODUCTION 

What happens to buildings during an earthquake? Well structural engineers have been studying this for years and they have discovered some interesting science and math is involved ……….. yes, you heard that correctly – MATH ! 

Today we are going to be structural engineers!
Buildings are complicated systems and their response to an earthquake can be complicated also. Here are some examples of some different ways a tall building can move or in other words, different modes:

Figure 7 – Computer Simulation of a Tall Building

Figure 8 - Swaying Back and Forth

Figure 9 - Swaying Side to Side

Figure 10 - Stretching and Compressing
Figure 11 - Sine Wave Shape

Figure 12 - Twisting
Before we can calculate the response of complicated buildings, let's look at a simple building. Take the following simplified 3 story building:

Here is the simplification of the building: We will explain how we are able to make this simplification later.

Here are some of the ways the building may move or deform (also known as the building's mode shape).
EXPLORATION

Part A-Let It Shake

**Materials:**
Each of the following is required per class:
- One large shaker table
- One 5-story structure

What do you observe? How does the building react at different frequencies? Focus on which floors move at each frequency and record your observations in the following table. Use the example drawings provided page 3.

<table>
<thead>
<tr>
<th></th>
<th>Comments</th>
<th>Sketches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Comments</td>
<td>Sketches</td>
</tr>
<tr>
<td>---</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>5 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 Hz</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Based on your data, what can you conclude about buildings shaken at different frequencies?

____________________________________________________________________________________
____________________________________________________________________________________
____________________________________________________________________________________

Before we can understand this phenomenon, we must review some math.
Part B – Math Background

Matrices are important mathematical tools that structural engineers use to solve many complex problems including what happens to buildings during earthquakes. As you have experienced, buildings react differently when they are shaken at different frequencies. Each of these different reactions is called a Mode.

Structural engineers use **matrix analysis** to determine the natural period of complex structures so that they can predict how a building will behave during an earthquake. Let’s investigate further matrix math!!!

### Introduction to Matrix Math

Right now, take a look around you. It may not seem like math is here, but it is.

For example, let’s create a simple chart. This chart will describe how many men and women there are in the room right now, and how many of them wear glasses.

<table>
<thead>
<tr>
<th></th>
<th>Glasses</th>
<th>No Glasses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Men</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Women</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Chart 1**

What if we wanted to know how many lenses the men had, and how many lenses the women had? We would take the number of men wearing glasses, multiply by two (number of lenses), and take the number of women wearing glasses and multiply by two.

What if I create a second chart which listed the number of lenses for people with glasses and people with no glasses?

<table>
<thead>
<tr>
<th>Number of Lenses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Glasses</strong></td>
</tr>
<tr>
<td><strong>No Glasses</strong></td>
</tr>
</tbody>
</table>

**Chart 2**

If I took the numbers in the first column (Labeled “Glasses”) of Chart 1 and multiplied each by two, I would have done the same thing. In mathematics, we call this type of chart, a **matrix**. When we “multiply” the charts together, we are actually doing matrix multiplication. The only difference is that we leave off the labels on the chart.
Rewrite your charts without the labels.

Chart 1

Chart 2

Taking a look at the men’s row first in Chart 1, we would want to multiply the number of men wearing glasses by two and the number of men not wearing glasses by 0. To do this we...

1. Take the first row, first column of Chart 1 and multiply it by the first row of Chart 2: ___
2. Take the first row, second column of Chart 1 and multiply it by the second row of Chart 2: ___
3. Add these two numbers and you have the number of lenses the men are wearing: ___
4. Put your answer in the Results Chart below.

Now we’ll repeat this with the women’s row.

1. Take the second row first column of Chart 1 and multiply it by the first row of Chart 2. ____
2. Take the second row second column of Chart 1 and multiply it by the second row Chart 2.____
3. Add these two numbers and you have the number of lenses the women are wearing. ___
4. Put your answer in the Results Chart below.

Our result will be a matrix (chart) of the form

\[
\begin{bmatrix}
\text{Men's Lenses} & \text{Women's Lenses}
\end{bmatrix}
\]

Results Chart

Putting this all together, our equation is (first matrix) times (second matrix) = result matrix

\[
\begin{bmatrix}
\text{First Matrix} \\
\text{Second Matrix}
\end{bmatrix} \times \begin{bmatrix}
\text{Second Matrix}
\end{bmatrix} = \begin{bmatrix}
\text{Result Matrix}
\end{bmatrix}
\]

Next we’re going to see why matrix multiplication is so useful and important.

Before we work an example, reflect on your observations of our multi-level mock structure on the shaker table. What are the two structural components of that structure? ______________ and ______________
CONCEPT DEVELOPMENT

In the previous section we discussed two important structural components of the multilevel mock structure. These components were __________ and __________. Let's take a look at some of the simplifications and assumptions engineers make when performing modal analysis. For this type of analysis the two most important features are a building's ________ and ________________.

Mass
To further simplify our problem, we assume that all the mass of a building is clumped into one spot on each of the floors and the roof (as indicated by the red dots in the Exploration); this makes the calculations more simple. To clarify, we will label our building with levels instead of floors and ceilings. For example, here is a two story building:

If we assume the mass of the building is located at level 1 and level 2. Then a sketch of the buildings mass would look like this:

Notice there is no mass for the ground floor (Level 0) because it sits directly on the ground.
Let's say level 2 has \( \frac{1}{2} \) the mass of level 1. Our Sketch would then look like this:

\[
\begin{align*}
m_1 &= 1 \text{ m} \\
m_2 &= \frac{1}{2} \text{ m}
\end{align*}
\]

The chart form of the mass matrix for this situation would look like this:

<table>
<thead>
<tr>
<th></th>
<th>Caused by level 1</th>
<th>Caused by level 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1 mass</td>
<td>( m_1 )</td>
<td>0</td>
</tr>
<tr>
<td>Level 2 mass</td>
<td>0</td>
<td>( m_2 )</td>
</tr>
</tbody>
</table>

The \( m_1 \) position describes the mass of level 1 caused by level 1. The \( m_2 \) position describes the mass of level 2 caused by level 2. The other two values are zero because the roof won’t affect the floor, and the floor won’t affect the roof.

The mathematical notation of the mass matrix for this situation would look like this:

\[
[M]= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}
\]

So our mass matrix becomes:

\[
[M]= \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}
\]
Stiffness

For purposes of determining the stiffness of the building, a sketch of the columns would look like this:

We have sketched and labeled two important components of a building, these are mass and stiffness. But, what exactly are mass and stiffness?

**Mass (m)** - In informal everyday usage, mass is more commonly referred to as weight, but in physics and engineering weight strictly means the size of the gravitational pull on the object; that is, how heavy it is, measured in units of force. Mass is independent of gravity. For example, a bowling ball would have a different weight on the moon, because the gravitation pull is different, but it would still have the same mass on the moon. For our example, we will focus on the mass of each floor of our mock structure.

**Force (P)** – A force is influence that may cause a body to accelerate. It may be experienced as a lift, a push, or a pull. A force may also cause rotation or deformation of the body. Earthquakes apply loads or forces to buildings. An earthquake shakes the ground on which a building sits. This shaking or movement has a displacement, velocity, and acceleration. From physics we know that: force = mass x acceleration or $F = ma$.

In the case of a building, the mass is related to the weight of the building and the acceleration comes from the earthquake and these two combine to create loads or forces on the building. Force is directly related to deflection (or deformation of the buildings) and this can cause a building to fail or break if it exceeds the strength of the building. For our example, we will focus on the resulting force from the shaker table shaking our mock structure.
Deflection (δ) - In engineering mechanics, deflection is a term that is used to describe the degree to which a structural element is displaced under a load. The deflection of a member under a load is directly related to the slope of the deflected shape of the member under that load and can be calculated by integrating the function that mathematically describes the slope of the member under that load. For our example, we will focus on the movement or deflection of our mock structure.

Stiffness (k) - Stiffness is the resistance of an elastic body to deflection or deformation by an applied force. It is an extensive material property. The stiffness k of a body that deflects a distance δ under an applied force P is

\[ k = \frac{F}{\delta} \]

Stiffness is very similar to modulus of elasticity. In our case, k defines the stiffness of the structure as a whole and includes the contributions of all the materials in the structure. For our example, we will focus on the stiffness of the metal columns of our mock structure.
Young's Modulus (or modulus of elasticity \( E \)) – In solid mechanics, Young's modulus (E) is the measure of the stiffness of a given material. It is also known as modulus of elasticity or tensile modulus. It is defined as the ratio of the change in stress over the change in strain. It is most important to realize it is defines the stiffness of a material or in other words, how hard a material is to deform. The modulus of elasticity for some common materials is listed in the following table. The higher the value the more force it takes to deform a material or the stiffer the material. For our example, we will focus on the type of material of the columns in our mock structure; this material is steel.

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s modulus (E) in lb/in^2 (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rubber</td>
<td>1,500-15,000</td>
</tr>
<tr>
<td>Nylon</td>
<td>150,000-435,000</td>
</tr>
<tr>
<td>Oak wood (along the grain)</td>
<td>1,600,000</td>
</tr>
<tr>
<td>High strength concrete (in compression)</td>
<td>6,500,000</td>
</tr>
<tr>
<td>Glass</td>
<td>10,400,000</td>
</tr>
<tr>
<td>Steel</td>
<td>30,000,000</td>
</tr>
<tr>
<td>Diamond</td>
<td>1500,000,000</td>
</tr>
</tbody>
</table>
Now after all of our simplifications for mass and stiffness, our simplified sketch of the building would look like this:

The lines on the bottom specify that the bottom of the building is attached to the ground.

Now we have to determine our stiffness matrix.

The stiffness matrix for a building is defined as:

\[
[K] = \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\]
In the preceding equations, we know that \( k \) represents the stiffness and \( E \) represents the Modulus of Elasticity of a column. But, what does \( I \) represent? \( I \) symbolizes the column’s moment of inertia, and it indicates the geometry of the column. Continue reading the next paragraph.

**Moment of Inertia (I)** - The moment of inertia of an object about a given axis describes how difficult it is to change its angular motion about that axis. For example, consider two discs of the same mass, one large and one small in radius. Assuming that there is uniform thickness and mass distribution, the larger radius disc requires more effort to accelerate it (i.e. change its angular motion) because its mass is effectively distributed further from its axis of rotation. Conversely, the smaller radius disc takes less effort to accelerate it because its mass is distributed closer to its axis of rotation. Quantitatively, the larger disc has a larger moment of inertia, whereas the smaller disc has a smaller moment of inertia.

So in short, moment of inertia is an object’s resistance to rotation or bending. For our example, we will focus on the shape of the columns of our mock structure to determine the mock structure’s moment of inertia.

\( I \) (moment of inertia) for a rectangular shape resisting rotation about its neutral axis (N.A.) (the neutral axis about which bending is occurring) is described in terms of the base dimension \( (b) \) and the height dimension \( (h) \):

\[
I = \frac{bh^3}{12}
\]

This can be demonstrated by comparing the strength of a 2” x 4” upright and on its side.

Record your observations by comparing the apparent strength of a 2” x 4” upright and on its side.

-------------------------------------------------------------------------------------------------

-------------------------------------------------------------------------------------------------

-------------------------------------------------------------------------------------------------
Equation of Motion

Up to this point we have only talked about stationary buildings, now let’s look at moving buildings. From equations of motion, we can derive a formula that relates stiffness, mass, natural frequency, and mode shapes.

\[ K \varphi_n = \omega_n^2 M \varphi_n \]

Where \( \omega_n \) is the angular natural frequency. Angular frequency (\( \omega \)) is the measure of how fast an item is spinning.

In our demonstration and example problem, nothing is spinning, but angular frequency is related to period (T) and frequency (f) as follows: \( \omega = \frac{2\pi}{T} = 2\pi f \)

If we were to take the way this circular motion changes and plot it on a straight line, then it is harmonic. The way buildings move back and forth can be thought of as harmonic; and therefore, we have to explore some terms that describe harmonic motions such as period (T) and frequency (f).

The period (T) is the amount of time it takes to complete one cycle.
In physics, resonance is the tendency of a system to oscillate at maximum amplitude at a certain frequency. This frequency is known as the system’s resonant frequency. When damping is small, the resonant frequency is approximately equal to the natural frequency ($\omega_n$) of the system, which is the frequency of free vibrations.

In matrix form this equation would be:

$$[K] \varphi_n = \omega_n^2 [M] \varphi_n$$

We can use this formula to solve for the mode shapes ($\varphi_n$) and natural frequencies ($\omega_n$).

CONCLUSION

1. From your observations in the Exploration, how do buildings respond to different frequencies?

2. From your observations and discussions, what are two major components of a structure that determines the structure’s response to an earthquake?

3. What mathematical tool can be used to determine how a building will respond during an earthquake? Briefly provide a description of this mathematical tool.
CONCEPT APPLICATION

Example:
A two story building is shown below. The mass of the building can be idealized as two lumps, one at the first level and one on the second level. The second level only weighs half as much as the first level. The earthquake shaking force is shown as \( P(t) \). The columns have height \( H \), modulus of elasticity \( E \), and moment of inertia \( I \).

\[
\begin{align*}
\text{P}_2(t) & \quad \text{m/2} \\
\text{P}_1(t) & \quad \text{m}
\end{align*}
\]

The mass matrix for this structure is:
\[
[M] = \begin{bmatrix}
m & 0 \\
0 & m/2
\end{bmatrix}
\]

\( m = 0.0095 \ lb \cdot s^2/\text{in} \)

\( b = 1 \text{ in} \)

\( h = 1/16 \text{ in} \)

\( H = 12 \text{ in} \)

\[
I = 2 \frac{bh^3}{12} = 2 \frac{(1\text{ in})(\frac{1}{16}\text{ in})^3}{12} = 0.00004069\text{in}^4
\]

The stiffness at level 1 is:
\[
k_1 = \frac{24(EL)}{H^3}
\]

The stiffness at level 2 is:
\[
k_2 = \frac{12(EL)}{H^3}
\]
Putting these two values into the global stiffness matrix:

\[
[K] = \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\]

In matrix form the equation of motion is:

\[
[K] \phi_n = \omega_n^2 [M] \phi_n
\]

Typically in algebra we could cancel the two \( \phi_n \), but when dealing with matrices they cannot cancel out. In order to solve this problem, we must first move everything to the left side of the equation.

\[
[K - \omega_n^2 M] \phi_n = 0
\]

\( \phi_n \) are not equal to zero. If they were equal to zero, then nothing would be happening with the building and what fun would that be?

So \( [K - \omega_n^2 M] \phi_n \) must equal zero. Because \( K \) and \( M \) are matrices we can’t just set \( K - \omega_n^2 M = 0 \).

Notice that \( [K - \omega_n^2 M] \) is also a matrix. What we do is set the determinant of this to equal zero.

\[
\text{det} [K - \omega_n^2 M] = 0
\]

The determinant is just a mathematical device we use to solve these types of equations. The determinant of a matrix is...

\[
\text{det} \begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix} = AD - BC
\]

Once we know the \( K \) and \( M \) matrices, we can plug them in and solve for \( \omega_n \).

\( \omega_n \) will help us determine at what frequencies the building will shake and \( \phi_n \) will show the shape.

We can use this formula to solve for the mode shapes \( (\phi_n) \) and natural frequencies following a series of steps as shown.
1. Find $[K - \omega_n^2 M]$

\[
\begin{bmatrix}
\frac{36EI}{H^3} & -\frac{12EI}{H^3} \\
-\frac{12EI}{H^3} & \frac{12EI}{H^3}
\end{bmatrix} - \begin{bmatrix}
\frac{m\omega_n^2}{H^3} & 0 \\
0 & \frac{1}{2} m\omega_n^2
\end{bmatrix} - \begin{bmatrix}
\frac{36EI}{H^3} & -\frac{12EI}{H^3} \\
-\frac{12EI}{H^3} & \frac{12EI}{H^3} - \frac{1}{2} m\omega_n^2
\end{bmatrix}
\]

2. Substitute $\alpha = \frac{12EI}{H^3}$ and $\beta = m\omega_n^2$

\[
\begin{bmatrix}
3\alpha - \beta & -\alpha \\
-\alpha & \alpha - \frac{1}{2} \beta
\end{bmatrix}
\]

3. Take the determinate and set it equal to 0:

\[
det \begin{bmatrix}
3\alpha - \beta & -\alpha \\
-\alpha & \alpha - \frac{1}{2} \beta
\end{bmatrix} =
\]

\[
(3\alpha - \beta) \left( \alpha - \frac{1}{2} \beta \right) - \alpha^2 = 0
\]

4. Multiply this out to get:

\[
\frac{1}{2} \beta^2 - \frac{5}{2} \alpha\beta + 2\alpha^2 = 0
\]

Multiply this equation by two to make it simpler.

$\beta^2 - 5\alpha\beta + 4\alpha^2 = 0$

We can solve this using the quadratic equation

\[
\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

In our case $a = 1, b = -5\alpha, c = 4\alpha^2$

\[
\beta = \frac{-(-5\alpha) \pm \sqrt{(-5\alpha)^2 - 4(1)(4\alpha^2)}}{2(1)} = \frac{5\alpha \pm \sqrt{25\alpha^2 - 16\alpha^2}}{2} = \frac{5\alpha \pm \sqrt{9\alpha^2}}{2} = \frac{(5 \pm 3)\alpha}{2}
\]

$\beta = 4\alpha$ and $\beta = 1\alpha$

5. Look at the first solution $\beta = 4\alpha$

Plug in $\alpha = \frac{12EI}{H^3}$ and $\beta = m\omega_n^2$

\[
m\omega_n^2 = 4 \cdot \frac{12(EI)}{H^3} = 48 \cdot \frac{EI}{H^3}
\]

6. Solve for $\omega_n = \sqrt{\frac{48}{mH^3} \left( \frac{EI}{H^3} \right)} = \sqrt{\frac{48 \left( \frac{30,000,000 \text{ lb}}{12 \text{ in}^3} \right) \left( 0.00004069 \text{ in}^4 \right)}{\left( 0.0993 \text{ lb} \cdot \text{in} \right)^2 \left( 12 \text{ in} \right)^3}} = 59.7 \frac{\text{rad}}{\text{s}}$
7. Look at the second solution \( \beta = 1\alpha \)

Plug in \( \alpha = \frac{12EI}{H^2} \) and \( \beta = m\omega^2 \)

\[
m\omega^2_{n2} = \frac{12EI}{H^3}
\]

8. Solve for \( \omega_{n2} \)

\[
\frac{12EI}{mH^2} = 3.750 \quad \sqrt{\frac{EI}{mH^2}} = \sqrt{\frac{12\left(30,000.000.000\frac{18}{in^2}\right)(0.00004069 \text{ in}^2)}{0.005\left(12 in^2\right)}} = 29.9 \text{ rad/s}
\]

Now that we have the natural circular frequencies \( (\omega_n) \), we can solve for the natural frequencies \( (f_n) \) and the natural periods \( (T_n) \).

<table>
<thead>
<tr>
<th>( f_n )</th>
<th>( \omega_n = \frac{rad}{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 = 59.7 \frac{rad}{s} )</td>
<td>( \omega_2 = 29.9 \frac{rad}{s} )</td>
</tr>
</tbody>
</table>

9. \( f_n = \frac{\omega_n}{2\pi} \frac{rad}{s} \)

\[9.5 \text{ Hz} = 9.5 \text{ Hz} \quad 4.8 \text{ Hz} = 4.8 \text{ Hz}\]

10. \( T_n = \frac{1}{f_n} \)

\[.11 \text{ s} \quad .21 \text{ s}\]

11. Each of the natural frequencies will have a particular mode shape associated with them.

For \( \omega_1 \) we have \( \phi_{n1} = \begin{bmatrix} \phi_{n11} \\ \phi_{n12} \end{bmatrix} \) for \( \omega_2 \) we have \( \phi_{n2} = \begin{bmatrix} \phi_{n21} \\ \phi_{n22} \end{bmatrix} \)

12. To find the mode shapes \( (\phi_n) \), individually substitute the \( \omega_n \) into the equation \([K - \omega_n^2 M][\phi_n] = 0\).

Substitute \( \omega_{n1} \) first.

\[
\begin{bmatrix}
\frac{36EI}{H^3} - m\omega^2_{n1} & -\frac{12EI}{H^3} \\
-\frac{12EI}{H^3} & \frac{12EI}{H^3} - \frac{1}{2} m\omega^2_{n1}
\end{bmatrix}
\begin{bmatrix}
\phi_{n11} \\
\phi_{n12}
\end{bmatrix}
= 0
\]

Remember that \( m\omega^2_{n1} = \frac{48EI}{H^3} \)

\[
\begin{bmatrix}
\frac{36EI}{H^3} - \frac{48EI}{H^3} & -\frac{12EI}{H^3} \\
-\frac{12EI}{H^3} & \frac{12EI}{H^3} - \frac{1}{2} \frac{48EI}{H^3}
\end{bmatrix}
\begin{bmatrix}
\phi_{n11} \\
\phi_{n12}
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
\frac{12EI}{H^3} & -\frac{12EI}{H^3} \\
-\frac{12EI}{H^3} & \frac{12EI}{H^3} - \frac{24EI}{H^3}
\end{bmatrix}
\begin{bmatrix}
\phi_{n11} \\
\phi_{n12}
\end{bmatrix}
= -\begin{bmatrix}
\frac{12EI}{H^3} \\
\frac{12EI}{H^3}
\end{bmatrix}
\begin{bmatrix}
\phi_{n11} \\
\phi_{n12}
\end{bmatrix}
= 0
\]

22
13. The \( \frac{12(EL)}{H^3} \) can be factored out since it is in each part of the matrix.

\[
- \frac{12(EL)}{H^3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0
\]

14. \( \frac{12(EL)}{H^2} \) is just a number and so we can divide both sides by it. Notice that \( m, E, I, \) and \( H \) have cancelled out. What matters is the relationship between the masses (here \( \frac{1}{2} \)) and the \( K \) matrix.

\[
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0
\]

15. Use the matrix multiplication you reviewed earlier to multiply the top row.

\[
-\phi_{11} - \phi_{12} = 0
\]

\[
-\phi_{11} = \phi_{12}
\]

If \( \phi_{11} = 1 \) then \( \phi_{12} = -1 \)

The bottom row would give you the same results so multiplying it out is not necessary.

16. This gives us \( \phi_{n1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) which looks like

17. Substitute \( \omega_{n2} \) next.

\[
\begin{bmatrix}
\frac{36(EL)}{H^3} - m\omega_{n2}^2 & -\frac{12(EL)}{H^3} \\
-\frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} - \frac{1}{2}m\omega_{n2}^2
\end{bmatrix}
\begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0
\]

Remember that \( m\omega_{n2}^2 = \frac{12(EL)}{H^2} \)

\[
\begin{bmatrix}
\frac{36(EL)}{H^3} - \frac{12(EL)}{H^3} & -\frac{12(EL)}{H^3} \\
-\frac{12(EL)}{H^3} & \frac{12(EL)}{H^3} - \frac{1}{2}\frac{12(EL)}{H^3}
\end{bmatrix}
\begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0
\]

\[
\begin{bmatrix}
\frac{24(EL)}{H^3} & -\frac{12(EL)}{H^3} \\
-\frac{12(EL)}{H^3} & \frac{6(EL)}{H^3}
\end{bmatrix}
\begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix}
\frac{24(EL)}{H^3} & -\frac{12(EL)}{H^3} \\
-\frac{12(EL)}{H^3} & \frac{6(EL)}{H^3}
\end{bmatrix}
\begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0
\]
18. The \( \frac{12(EI)}{H^3} \) can be factored out since it is in each part of the matrix.

\[
\frac{12(EI)}{H^3} \begin{bmatrix} 2 & -1 \\ -1 & 0.5 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0
\]

19. \( \frac{12(EI)}{H^3} \) is just a number and so we can divide both sides by it.

\[
\begin{bmatrix} 2 & -1 \\ -1 & 0.5 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0
\]

20. Use the matrix multiplication you reviewed earlier to multiply the top row.

\[
2\phi_{21} - \phi_{22} = 0
\]

\[
2\phi_{21} = \phi_{22}
\]

If \( \phi_{21} = 1 \) then \( \phi_{22} = 2 \).

Again, the bottom row would give you the same results so it’s not necessary.

21. This gives us \( \phi_{n2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) which looks like

![Diagram](image-url)
APPLICATION PROBLEM

Given the following structure (with the same K-matrix as our earlier example):

\[ m = 0.0095 \, \text{lb} \cdot \text{s}^2 / \text{in} \]
\[ m_1 = m_2 = m \]

The mass matrix for this structure is:

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

The k matrix for this structure is:

\[
\begin{bmatrix}
\end{bmatrix}
\]

B = 1\,\text{in}
\[ h = 1/16 \, \text{in} \]
\[ H = 12\,\text{in} \]

Look up (in this activity) any other quantities that you need.

The structure is made out of steel.

When you are finished, build a model to these specs and shake the model at the natural frequencies and see what happens.
1. Find \([K - \omega_n^2 M]\)

2. Substitute \(\alpha = \frac{12EI}{h^3}\) and \(\beta = m\omega_n^2\)

3. Take the determinate and set it equal to 0:

4. Multiply this out to get:

We can solve this using the quadratic equation

\[
\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

In our case \(a = \ldots, b = \ldots, c = \ldots\)

5. Look at the first solution \(\beta = \ldots\)

Plug in \(\alpha = \frac{12EI}{h^3}\) and \(\beta = m\omega_n^2\)

6. Solve for \(\omega_{n1} = \ldots\)
7. Look at the second solution $\beta =$

Plug in $\alpha = \frac{12(El)}{H^2}$ and $\beta = m\omega_n^2$

8. Solve for $\omega_{n2} =$

Now that we have the natural circular frequencies ($\omega_n$), we can solve for the natural frequencies ($f_n$) and the natural periods ($T_n$).

<table>
<thead>
<tr>
<th>$\omega_{n1} =$</th>
<th>$\omega_{n2} =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_n = \frac{\omega_n}{2\pi \text{ rad}} =$</td>
<td></td>
</tr>
<tr>
<td>$T_n = \frac{1}{f_n} =$</td>
<td></td>
</tr>
</tbody>
</table>

9. Each of the natural frequencies will have a particular mode shape associated with them.

For $\omega_{n1}$ we have $\phi_{n1} = [\phi_{11} \phi_{12}]$, for $\omega_{n2}$ we have $\phi_{n2} = [\phi_{21} \phi_{22}]$.

12. To find the mode shapes ($\phi_n$), individually substitute the $\omega_n$ into the equation $[K - \omega_n^2M][\phi_n] = 0$.

Substitute $\omega_{n1}$ first.

$$\begin{bmatrix} \frac{36(El)}{H^3} - m\omega_{n1}^2 & -\frac{12(El)}{H^3} \\ -\frac{12(El)}{H^3} & \frac{12(El)}{H^3} - m\omega_{n1}^2 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0$$
13. The $\frac{-12EI}{H^2}$ can be factored out since it is in each part of the matrix.

14. $\frac{12EI}{H^2}$ is just a number and so we can divide both sides by it. Notice that $m$, $E$, $I$, and $H$ have cancelled out. What matters is the relationship between the masses (here 1) and the K matrix.

15. Use the matrix multiplication you reviewed earlier to multiply the top row.

If $\phi_{11} = 1$ then $\phi_{12} =$

The bottom row would give you the same results so multiplying it out is not necessary.

16. This gives us

$$\phi_{n1} = \begin{bmatrix} 1 \end{bmatrix}$$

which looks like

17. Substitute $\omega_{n2}$ next.

$$\begin{bmatrix} \frac{36EI}{H^3} - m\omega_{n2}^2 & \frac{-12EI}{H^2} \\ \frac{-12EI}{H^3} & \frac{12EI}{H^3} - m\omega_{n2}^2 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 0$$
18. The \( \frac{12}{H^2} \) can be factored out since it is in each part of the matrix.

19. \( \frac{12}{H^2} \) is just a number and so we can divide both sides by it.

20. Use the matrix multiplication you reviewed earlier to multiply the top row.

If \( \phi_{21} = 1 \) then \( \phi_{22} = \)

Again, the bottom row would give you the same results so multiplying it out is not necessary.

21. This gives us

\[
\phi_{n2} = \begin{bmatrix} 1 \end{bmatrix}
\]

which looks like

What did you learn about a building’s response to an earthquake?
WEBQUEST

Advanced Earthquake Topics

Introduction
Earthquake engineering is a broad field and we have only covered one small element. We haven’t even talked about what makes earthquakes happen, how earthquakes differ, infamous earthquakes, or how we measure the size of an earthquake. Let’s take some time, and learn a little more about earthquakes.

The Task
Pick one of the following questions to research online. Your response should be as complete as possible and include the references you used.

1. What is the Richter scale and how does it relate to earthquakes?
2. What is resonance? Can you find an example of a structural failure by resonance during an earthquake?
3. What is the significance of the Old Tacoma Narrows bridge failure?
4. Describe the following earthquakes:
   - 1985 Mexico City
   - 1994 North Ridge California
   - 1812 New Madrid, Missouri
5. Where and when was the largest recorded earthquake? The deadliest? The most costly ($)?
6. What is meant by “the ring of fire”?
7. What is the relationship between volcanoes and earthquakes?
8. How does soil type affect the results of earthquakes?
The Process

1. Pick one of the eight questions to answer and explore. Get teacher approval before continuing.
2. Research your question on-line. Try searching for the answers using the following: Google, Alta Vista, Wikipedia, or the USGS web-site.
3. After you have researched and found an answer or answers to your question, write what you have learned and provide a list of references used.
4. When everyone is finished, we will share our answers.

Evaluation

Use the following table to determine the level of knowledge you gained:

<table>
<thead>
<tr>
<th></th>
<th>Beginning</th>
<th>Developing</th>
<th>Accomplished</th>
<th>Exemplary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stated Objective</td>
<td>A general/short answer to the question</td>
<td>A general/short answer to the question and a few other small facts.</td>
<td>A complete answer to the question.</td>
<td>A complete answer to the question and extensive background information and possibly purposing a new questions.</td>
</tr>
</tbody>
</table>

Conclusion

By answering one of the assigned questions, you have explored one more aspect of earthquake engineering. You can apply this same process to any field or topic you wish to explore. Simply think of a question you would like to know the answer and then find the answer, or even better find the answer and a new question!