Classical Mechanics and Statistical Mechanics/Thermodynamics
August 2021

Please adhere to the following:

- Please use only the blank answer paper provided.
- Please use only the reference material supplied.
- Please use only one side of the answer paper.
- Please put your alias (and NOT your real name) on every page.
- After you have completed a problem, put three numbers on every page used for that problem:
  1. The first number is the problem number.
  2. The second number is the page number for that problem (please start each problem with page number "1").
  3. The third number is the total number of pages you used to answer that problem.
- Please do not staple your exam nor the individual problems.

Problem 1:

(a) (3 points) A pool cue hits a billiard ball with an impulse J by applying a horizontal force through the center of mass of the billiard ball. If the billiard ball with mass M and radius R has a coefficient of friction between the ball and the table of μ, how much time t_{elapse} elapses before the ball starts to roll without slipping? You can assume that the ball is a solid sphere with moment of inertia I,

$$I = \frac{2}{5}MR^2, \tag{1}$$

and you can neglect friction during the time the impulse is applied.

(b) (1 point) How far does the billiard ball travel during the time t_{elapse}?

(c) (1 point) What fraction of the billiard ball's energy was lost while it was sliding before it started to roll without slipping?

(d) (1 point) Is it possible to apply a horizontal impulse to a stationary billiard ball in such a way that the billiard ball would roll without slipping from the moment it is hit? If so, explain qualitatively where the force should be applied on the billiard ball and why.

(e) (3 points) Calculate the distance from the center of mass of the billiard ball where the force should be applied according to your answer in part (d).

(f) (1 point) An ideal ball described in undergraduate physics classes will roll with no friction and never slow down but a real ball does slow down and does eventually stop rolling. Explain why this happens.

Problem 2:

Similarly to an isotropic three-dimensional harmonic oscillator, an electron with mass m_e in a "graded semi-conductor sphere" (the particulars of the device do not matter for the purpose of this problem) is characterized by the potential energy $V(r)$,

$$V(r) = V_0 r, \tag{2}$$

where r is the radial distance from the center of the graded semiconductor sphere and V_0 is a positive constant. In this problem you should treat the electron as a classical particle.

(a) (1 point) Calculate the restoring force.

(b) (1.5 points) Write the Lagrangian of this problem in spherical coordinates.

(c) (2 points) Draw a schematic of the equivalent one-dimensional effective potential as a function of r for two different cases: (i) $l = 0$ and (ii) $l \neq 0$. Here, l denotes the magnitude of the angular momentum vector.

(d) (3 points) Calculate the radius of the electron orbit as a function of the angular momentum l for the case that the orbit is circular.

(e) (2.5 points) If the orbit deviates slightly from being circular, what is the frequency of the small oscillation in the radial distance?

Problem 3:

A particle of charge e and mass m (position vector $\vec{r}$, with components x, y, and z) moves in an electric field given by the vector potential $\vec{A}(\vec{r},t)$ and scalar potential $\Phi(\vec{r},t)$,

$$\vec{A}(\vec{r},t) = -By\,\hat{x} \tag{3}$$

and

$$\Phi(\vec{r},t) = 0, \tag{4}$$

where B is a constant and $\hat{x}$ denotes the unit vector along the positive x direction. For an arbitrary field, the Lagrangian in cgs units is given by

$$L = \frac{1}{2}m\,(\dot{\vec{r}}(t))^2 - e\,\Phi(\vec{r},t) + \frac{e}{c}\,\dot{\vec{r}}(t)\cdot\vec{A}(\vec{r},t). \tag{5}$$

(a) (2 points) Find the Euler-Lagrange equations of motion in Cartesian coordinates.

(b) (1.5 points) Show that your result from part (a) is consistent with the Lorentz force law.

(c) (3.5 points) Find expressions for $x(t)$ and $y(t)$ and show that they are periodic. Find the angular frequency and show that they differ by a fixed phase.

(d) (3 points) Calculate the action variables J_x and J_y.

Problem 4:

A problem on gas expansion.

(a) (2 points) A physics student performed the following experiment on a gas of n moles of an ideal gas contained in a thermally isolated box that had an initial temperature T: The gas was initially confined in a volume V_1 that was separated from another vacant volume V_2 by a partition. The partition was suddenly removed. The gas underwent free expansion to the volume $V_1 + V_2$, while the whole system was kept thermally isolated. It was found that the final temperature T_f was equal to the initial temperature T. This was found to be true for any T, V_1, and V_2.

From this experiment, what conclusion can we draw regarding the internal energy $U(T, V)$ of the ideal gas as a function of temperature T and volume V?

(b) (3 points) With the same initial condition as in part (a), the student now allowed the system to be in thermal contact with a reservoir at temperature T, while they slowly moved the partition to have the same final volume $V_1 + V_2$ as in part (a).

Find the change of the entropy of the ideal gas between the initial and final states. Assume the ideal gas equation of state with the gas constant R.

(c) (4 points) Suppose that the heat capacity C_V at constant volume of the system was found to be $C_V = nRT^\gamma$ with a constant $\gamma > 0$. With the same initial condition as in part (a), the student now moved the partition slowly to the final volume $V_1 + V_2$, while keeping the system thermally isolated.

Find the final temperature of the system.

(d) (1 point) If a non-ideal gas was allowed to expand as in part (a), would its temperature increase or decrease? Explain your reasoning.

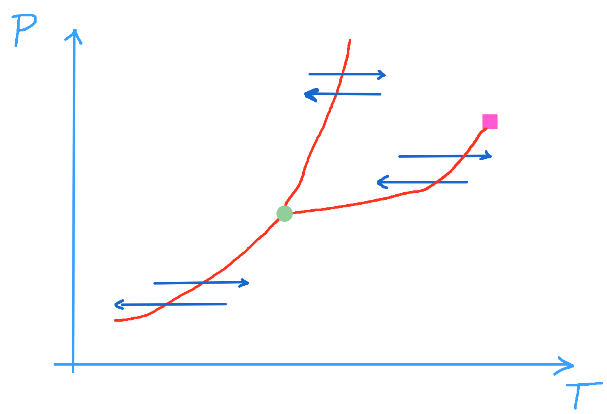

Figure 1: Schematic for Problem 5.

Problem 5:

(a) Consider the PT-diagram shown above.

- (ai) (1 point) Label the phases.
- (aii) (2 points) Label each of the arrows (i.e., give each of the arrows a name).
- (aiii) (1/2 points) What is the point marked by the circle called?
- (aiv) (1/2 points) What is the point marked by the square called?

(b) (3 points) Denote the temperature and pressure marked by the square by (T^*, P^*). Consider a PV-diagram and draw isotherms for $T = T^*$, $T > T^*$, and $T < T^*$.

(c) (3 points) Describe in words what the isotherms in the PV-diagram tell you.

Problem 6:

Consider a collection of non-interacting identical bosons in a two dimensional (2D) isotropic harmonic trap, characterized by the quantum mechanical single-particle Hamiltonian $\hat{H}_{\text{sp}}$,

$$\hat{H}_{\text{sp}} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}m\omega^2\hat{y}^2. \tag{6}$$

The single-particle energies are $E_{\mathbf{m}} = \hbar\omega(m_x + m_y + 1)$, where $\mathbf{m} = (m_x, m_y)$ with $m_{x,y} = 0, 1, 2, \cdots$. In the following questions you may neglect the zero point energy for simplicity, i.e., you may use $E_{\mathbf{m}} = \hbar\omega(m_x + m_y)$.

(a) (2 points) Using the fact that bosons have no restriction on the occupancy of a single quantum state, derive the average thermal occupation $\langle\hat{n}_{\mathbf{m}}\rangle$ of a single level $\mathbf{m}$ of the harmonic trap within the grand-canonical ensemble. You should find

$$\langle\hat{n}_{\mathbf{m}}\rangle = \frac{1}{e^{\beta(E_{\mathbf{m}}-\mu)} - 1}, \tag{7}$$

where μ denotes the chemical potential. Explicitly state any assumption you have made about the chemical potential μ and show that the assumption is self-consistent.
Hint: You may find the following identity useful:

$$\sum_{l=0}^{k} x^l = \frac{1 - x^{k+1}}{1 - x}. \tag{8}$$

(b) (2 points) Show that the density of single-particle states $D(E)$ for the 2D harmonic potential is

$$D(E) = \frac{E}{\hbar^2\omega^2}. \tag{9}$$

(c) (4 points) Using the density of states derived in (b), compute the critical temperature T_c below which a Bose-Einstein condensate forms. You should find

$$T_c = \frac{\sqrt{6}\hbar\omega}{\pi k_B}\sqrt{N}, \tag{10}$$

where k_B denotes the Boltzmann constant and N the mean total number of particles. In addition, show that the condensate fraction N_0/N can be written as

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^2, \tag{11}$$

where N_0 is the mean number of atoms in the ground-state of the harmonic potential. Explain the key steps of your calculation and justify any assumptions you make.
Hint: You may find the following integral useful:

$$\int_0^\infty \frac{x}{e^x - 1}dx = \frac{\pi^2}{6} \tag{12}$$

(d) (2 points) If the gas was instead confined in a 2D box of area $A = L^2$, would you expect a Bose-Einstein condensate to form at any temperature? Comment on any differences with the calculation in parts (a)-(c) for a harmonic confining potential. Note: It will be sufficient to analyze the density of single-particle states $D(E)$ and to draw conclusions based on the behavior of $D(E)$; you do **not** have to repeat the full calculation.