Classical Mechanics and Statistical/Thermodynamics

August 2024

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.
7. **There are 5 problems but only 4 problems will count to your grade. If you choose to solve all 5, the problem on which you score the least will be discarded. Please attempt at least four problems as partial credit will be given.**

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \frac{\pi^2}{6}$$

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\ dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{or} \quad \log(n!) \approx n\log(n) - n$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Handy Taylor Series:

$$\log(1+x) = \sum_{n=1}^\infty (-1)^{n+1} \frac{x^n}{n}$$

$$\log(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$
$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$
$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12} \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 6.63 \times 10^{-34}$m^{2}kg/s
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm} \cdot \mathrm{mol}^{-1}\mathrm{K}^{-1}$

Question 1: A cylindrical disk with mass m, radius R and height h sits on a frictionless, horizontal, flat surface. The disk has a thin, massless thread wrapped around its circumference. The disk is initially at rest when a force F is applied to the thread, pulling it tangentially, causing the disk to accelerate as the thread unwinds from the disk. Gravity acts perpendicular to the surface.

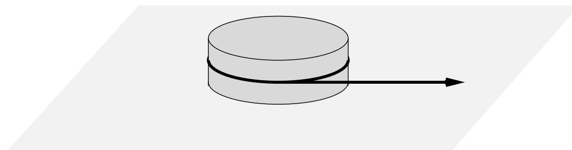

Figure 1: The disk and thread discussed above.

(a) Show that the moment of inertia of the disk about an axis through the center of the disk and perpendicular to the plane is given by $I = mR^2/2$. (2 points)

(b) Determine the angular velocity, ω, of the disk when a length L of thread has been unwound from the disk. Give your answer in terms of L, m, F and R. (2 points)

(c) Determine the amount of time required for the applied force to unwind a length L of thread from the disk. (2 points)

(d) Determine the linear velocity of the disk at that time. (2 points)

(e) Given an expression for the total kinetic energy of the disk as a function of L, the length of thread unwound from the disk by the applied force. (2 points)

Question 2: Two masses M_A and M_B are attached to a trio of identical springs with elastic constant K inside a frictionless tube of length L, as shown in Fig. 2. Assume that $M_A > M_B$ throughout this problem. For parts (a)-(d), consider a scenario where the tube is placed horizontally on top of a flat table [see Fig. 2(a)].

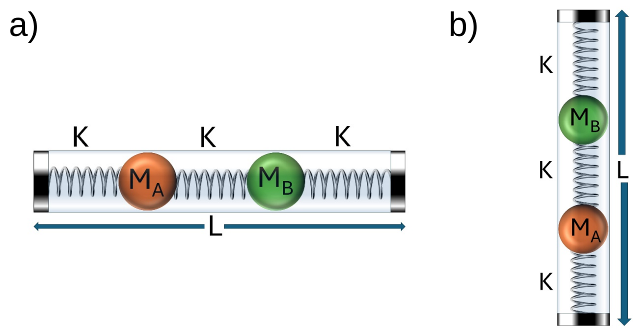

Figure 2: a) Horizontal configuration of apparatus for questions (a)-(d). b) Vertical configuration of apparatus for question (e). For all problems, gravity acts along the vertical direction.

(a) Calculate the equilibrium position of each of the masses. (1 point)

(b) Obtain the equations of motion for the position of each mass. (2 points)

(c) Solve the equations of motion you obtained in (b). Specifically, obtain the frequencies of the normal modes describing the collective motion of the masses. (2 points)

(d) Describe the motion of the masses given by your solution to part (c). In particular, computing the normal co-ordinates associated with the normal modes in your solution. Do you identify any conserved quantities in the problem? If so, what are they? (1 point)

(e) Consider now the situation where the tube is placed *vertically* on a table, as shown in Fig. 2(b). Discuss how this will change your answers to each of the questions (a)-(d) (2 points)

Question 3: In June 2023, the Titan submersible imploded at a depth of about 4000 m below the surface of the ocean. Some news reports claimed the violent adiabatic compression would have created temperatures hotter than the surface of the sun. Fully modeling this rapid and irreversible process is not a qualifier problem. However, we can do a back of the envelope calculation using reversible thermodynamic processes.

A volume of nitrogen V_1 and temperature T_1 is at the surface of the ocean at atmospheric pressure (P_1). For this thought experiment, consider the gas to be in a cylinder with a piston. Initially, the piston is clamped so that it cannot move. The cylinder with the gas is lowered to depth d in sea water with density ρ. Ignore any changes in temperature with depth so that this ocean is at uniform temperature T_1. When the cylinder is at depth d, the piston is unclamped and the gas undergoes adiabatic compression to V_2 and T_2. The cylinder remains at depth d with the piston free to move as the gas cools isobarically to V_3 and T_1.

(a) What is the pressure at depth d below the surface of the ocean? (1 points)

(b) *Adiabatic compression:* Find the volume V_2 and the temperature T_2, immediately after the adiabatic compression. Also compute the entropy change ΔS for this process. (3 points)

(c) *Isobaric cooling:* Find the volume V_3 after the gas cools to T_1, immediately after the adiabatic compression. Also compute the entropy change ΔS for this process. (3 points)

(d) If the process was repeated using a volume of argon, would the temperature change between T_1 and T_2 be different? If so, would the change be greater or smaller compared to nitrogen? Explain your reasoning. (1 point)

(e) The heat capacity we typically use for diatomic molecules at "normal temperatures" accounts for energy in the three translational and two rotational degrees of freedom. At sufficiently high temperatures, there are additional internal degrees of freedom that should be considered. In the case of nitrogen, molecular vibrations will contribute to the heat capacity. How would the inclusion of vibrational degrees of freedom affect your answers to parts (b) and (c). I.e., will V, T and ΔS remain the same, increase or decrease? Explain your answers. (2 points)

Question 4:

Consider a physical system composed of N distinguishable spins assuming two possible values ± 1. The two values the spin may take correspond to a pair of energy levels with energies $\pm\epsilon$, respectively. We can study the statistical mechanics of this system using the microcanonical or canonical ensembles.

(a) Let's first try to use the microcanonical ensemble. Assuming the total system has energy E, find the number of different possible configurations of the spins. Express your answer in terms of E and N. (1 point)

(b) Show that the entropy of the system in the limit $N \gg 1$ is,

$$S(E,N) \approx k_B \left[N\log N + N\log 2 - \frac{1}{2}\left(N + \frac{E}{\epsilon}\right)\log\left(N + \frac{E}{\epsilon}\right) - \frac{1}{2}\left(N - \frac{E}{\epsilon}\right)\log\left(N - \frac{E}{\epsilon}\right)\right].$$

(2 points)

(c) Obtain an expression for the total internal energy as a function of the number of spins and temperature, $E(N,T)$. (2 points)

(d) Now, we will try to find the same quantities using the canonical ensemble. Show that the partition function for the total system can be expressed as,

$$Q(T,N) = 2^N \left[\cosh\left(\frac{\epsilon}{k_B T}\right)\right]^N .$$

(1 point)

(e) Compute the average internal energy $U(T,N)$ for the canonical ensemble. (2 points)

(f) Compare your result for E for the microcanonical ensemble with the results for U for the canonical ensemble. Should they agree? What is the difference in the meaning of E for the microcanonical ensemble and U for the canonical ensemble? (2 points)

Question 5: When particles have kinetic energy $E \gg mc^2$, with m the rest mass, they must be treated as relativistic with single-particle kinetic energy given by $E_{\mathbf{p}} \simeq |\mathbf{p}|c$. Consider a gas of N relativistic electrons confined to a volume $V = L^3$.

(a) Show that the Fermi energy for the system is,

$$E_F = \hbar \pi c \left(\frac{3N}{\pi V}\right)^{1/3}.$$

(2 points)

(b) Using the result from part (a) show that the ground-state energy for a relativistic electron gas at zero temperature can be expressed as,

$$U = \frac{3}{4} N E_F.$$

(2 points)

(c) Obtain an expression for the chemical potential $\mu(T)$ of the electron gas in the low temperature limit ($T \ll T_F$). Include finite temperature contributions to $\mathcal{O}(T^2)$. Hints: i) You may find the following low temperature approximation a useful starting point,

$$\int_0^\infty \frac{H(\epsilon)}{e^{\beta(\epsilon-\mu)}+1}\, d\epsilon \approx \int_0^\mu H(\epsilon)\, d\epsilon + \frac{\pi^2}{6\beta^2} \left.\frac{dH}{d\epsilon}\right|_\mu,$$

where $H(\epsilon)$ is an arbitrary function. ii) Consider the value of the chemical potential at $T = 0$. Use this to further approximate the integral in the RHS of the above equation. (3 points)

(d) Obtain the scaling of the heat capacity of the relativistic gas with temperature, i.e., $C_V \propto T^\alpha$ where α is some numerical constant, in the limit $T \ll T_F$. Is your result consistent with the expectations of the classical equipartition theorem? Explain why it is/isn't different. Hint: You can compute the heat capacity exactly by exploiting your result from part (c) or approximately by using physical arguments. Either approach is acceptable for this question. (3 points)