Classical Mechanics and Statistical/Thermodynamics

August 2025

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\,\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 6.63 \times 10^{-34}$m^{2}kg/s
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

1. A box of mass m rests on the top surface of a wedge of mass M. All surfaces are frictionless, including the ground, so that both the box and the wedge are free to move. The upper edge of the wedge makes an angle θ relative to the ground. The mass is a distance L from the bottom of the wedge (as measured along the surface). Let x be the horizontal position of the point where the top wedge surface meets the ground.

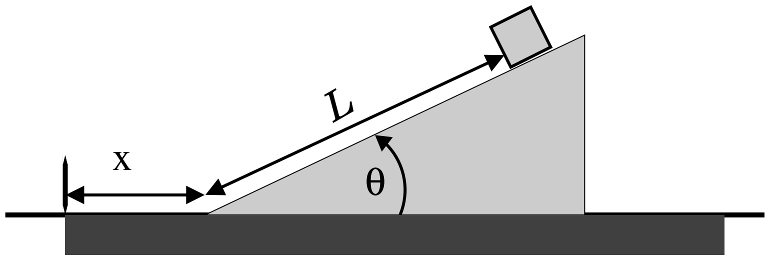

Figure 1: The wedge and the small cube slide without friction.

(a) (2 points) Find the lagrangian for the system using x and L as generalized coordinates.

(b) (2 points) Find the Euler-Lagrange equations for the system.

(c) (2 points) Find the equation of motion for the box relative to the surface of the wedge. Verify that the limit of $M \to \infty$ makes sense.

(d) (2 points) Use part (c) to find how long it takes for the box to reach the bottom, assuming that the box and wedge start at rest, and that $L(t = 0) = L_0$.

(e) (2 points) Verify that your answers to (d) make senste in the limits of $\theta \to 0$ and $\theta \to \pi/2$.

2. A thin uniform rod of length L and mass M can rotate without friction around a pivot point at one end. It is initially at rest, held in place horizontally. A small ball of mass m is attached to the same pivot point by a massless thread also of length L. It also is intially at rest, hanging immediately below the pivot. In the figure, gravity points downward, with a magnitude g.

   The rod is released and swings around the pivot point to collide with the ball. Assume that we can treat the pivot and rod to be of negligible width.

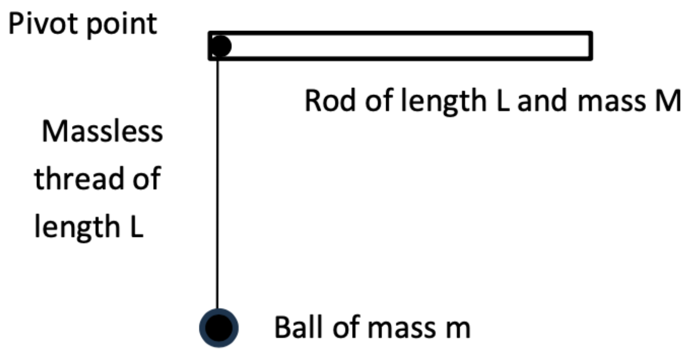

Figure 2: All objects start from rest. Gravity points down the page.

   (a) (1 point) Show that the moment of inertia of a thin uniform rod of mass M and length L rotating around one end is given by $(1/3)ML^2$.

   (b) (1 point) If the rod is released from rest what is its angular speed just before colliding with the ball? (Ignore the finite radius of the ball, and treat the collision as occurring when the rod points vertically downward).

   (c) (1 point) The rod collides elastically with the ball. It is found that after the collision, the ball swings upward and circles around the pivot. What is the *minimum* speed the ball can have immediately after the collision in order for it to be able to do so?

   (d) (3 points) What is the ratio of the masses m/M that will give the ball this minimum speed?

   (e) (1 point) What is the angular velocity of the rod after the elastic collision using the answer you found in (c)? (If you cannot solve (c), write down the answer for the general case.)

   (f) (3 points) Return to the general case of where m/M is arbitrary. If the rod is released from rest and hits the same ball, but the collision is completely *inelastic*, how high will the ball rise after the collision?

3. Two identical point-like objects, each of mass m, hang from strings each of length L. The masses are coupled by an ideal, massless spring with spring constant k, and its equilibrium length (when neither stretched nor compressed) b is equal to the distance between the strings' supports.

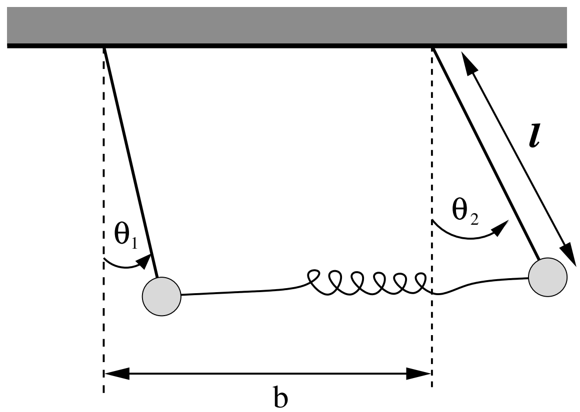

Figure 3: Gravity points down the page.

Throughout this problem you should use the approximation of small oscillations.

(a) (2 points) Find the lagrangian L in terms of the generalized coordinates θ_1 and θ_2.

(b) (2 points) Find the normal co-ordinates Q_1 and Q_2 such that the Lagrangian has the form

$$L = \sum_{j=1}^{2} \frac{1}{2}\left(\dot{Q}_j^2 - \omega_j^2 Q_j^2\right)$$

Express the relation between the normal coordinates and the original coordinates as $Q_j = M_{j,k}\,\theta_k$ where $\mathbf{M}$ is a matrix.

(c) (1 point) Find the frequencies ω_1 and ω_2 of the normal modes.

(d) (2 points) Let p_j denote the canonical momentum conjugate to q_j, and let P_j denote the canonical momentum conjugate to to Q_j. Find the relation between the two sets of momenta and express it in terms of the matrix $\mathbf{M}$ found in part (b).

(e) (3 points) Find a generating function $F_2(\theta, P)$ for the canonical transformation from the original coordinates and momenta to the normal coordinates and momenta.

4. Consider a system consisting of N non-interacting, distinguishable particles at temperature T. Each particle is a spin 1 system with the Hamiltonian

$$H_1 = -\mu_0 B \sum_{i=1}^{N} s_i,$$

with $s_i \in \{-1, 0, 1\}$. In this expressions B is the applied magnetic field, and μ_0 is the Bohr magneton.

(a) (2 points) Calculate $\bar{M}$, the average magnetization of the system,

$$\bar{M} \equiv \langle \mu_0 \sum_i s_i \rangle \tag{1}$$

and $\langle \ldots \rangle$ denotes the thermal average.

(b) (1.5 points) Determine $\bar{M}$ to leading (i.e. non-zero) order in the high temperature limit, and explain the physics behind this value.

(c) (1.5 points) Find $\bar{M}$ in the limit of low temperature, and explain the physics behind this value.

(d) (2 points) Calculate χ, the susceptibility of the system.

(e) (1.5 points) Determine χ in the high temperature limit, and explain the physics behind this value.

(f) (1.5 points) Find χ to leading (i.e. non-zero) order in the limit of low temperature, and explain the physics behind this value.

5. Consider an ideal quantum gas of N particles each with a mass m in a volume $V = L^3$ in three dimensions.

   (a) (2 points) Show that the density of states in energy, $\mathcal{D}(E)$ can be written as:

   $$\mathcal{D}(E) = \alpha\, E^{1/2}$$

   where α depends upon physical and/or mathematical constants, and determine the value of α.

   (b) First consider the gas as if it consisted of spinless fermions (or simply fully polarized spin-1/2 particles) and treat it in the grand canonical ensemble.

      i. (2 points) Using the density of states and the standard form for the Fermi-Dirac distribution, show that the average number of particles $\bar{N}_f(T, V, \mu)$ is given by

      $$\bar{N}_f(T, V, \mu) = \frac{V}{\lambda^3} f_{3/2}(z)$$

      where $\lambda \equiv h/\sqrt{2\pi m k_B T}$ is the thermal de Broglie wavelength, $z \equiv e^{\mu/k_B T}$ and the function $f_p(z)$ is defined on page 2 of the exam. (If you can't find α in part (a), leave it as an undetermined constant).

      ii. (2 points) For large z, the function $f_{3/2}(z)$ can be approximated as:

      $$f_{3/2}(z) \approx \frac{4}{3\sqrt{\pi}} (\log z)^{3/2} .$$

      Use this to show that at low temperature $\bar{N}_f(T, V, \mu)$ becomes independent of temperature, and invert your answer to find $\mu(\bar{N}, V)$ in this low temperature limit and explain its physical meaning.

   (c) Finally treat the gas as spinless bosons using the grand canonical ensemble.

      i. (2 points) Using the density of states and the standard form for the Bose-Einstein distribution, show that naively integrating over energy gives that the average number of particles $\bar{N}_b(T, V, \mu)$ is given by

      $$\bar{N}_b(T, V, \mu) = \frac{V}{\lambda^3} g_{3/2}(z)$$

      where the function $g_p(z)$ is defined on page 2 of the exam. (If you can't find α in part (a), leave it as an undetermined constant).

      ii. (2 points) The function $g_{3/2}(z)$ has a maximum value ≈ 2.61 when $z = 1$. The implication is that at low temperatures $\mu \approx 0$ and there is a maximum number of bosons allowed in a given volume in space.
      Either explain why this is the case, or explain what went wrong in this calculation of $\bar{N}_b(T, V, \mu)$, and how one can fix it. In either explanation describe the physics behind the mathematics.