Classical Mechanics and Statistical/Thermodynamics

January 2007

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$

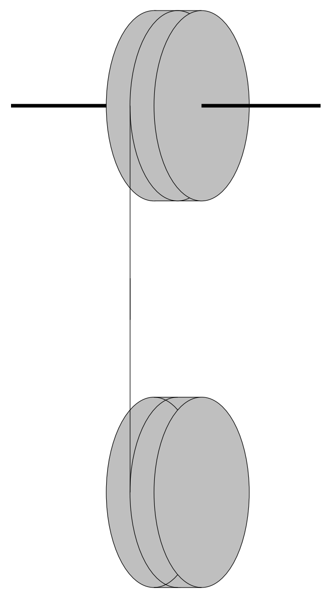
$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$

$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

Classical Mechanics

1. A uniform disk of mass M and radius R is attached to a frictionless axle, so that it can spin, but not otherwise move. A string of negligible mass is wrapped around the disk and then wrapped around a second disk also of mass M and radius R. The system starts from rest, and the second disk is released so that it accelerates downward **and** starts to spin.



   (a) Draw a clear, free-body diagram for the system, labelling all forces. Write down Newton's second laws for each part of the system. **(1 point)**

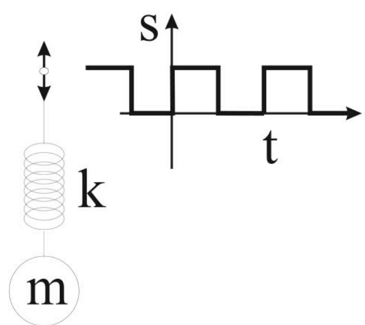
   (b) Solve for the acceleration in the vertical direction of the second disk. **(3 points)**

   (c) Solve for the tension in the string connecting the disks. **(2 points)**

   (d) Using conservation of energy, calculate the vertical speed of the falling disk after it has fallen a distance L. **(4 points)**

2. An object of mass, m, hangs on a spring of constant k. The upper end of the spring is moved up and down according to a periodic square wave function as shown. The square wave function may be written:

$$s(t) = 1 \qquad 0 < t < \frac{T}{2}$$
$$s(t) = 0 \qquad \frac{T}{2} < t < T$$



(a) What is the Newtons second law equation for the system? **(2 Points)**

(b) What is the Fourier series respresentation of the driving force for the system? **(3 Points)**

(c) What is the steady state solution for the displacement of the system? **(3 Points)**

(d) If the resonant frequency of the mass-spring system is ω_0 and the period of the driving force equals $\frac{6\pi}{\omega_0}$, what term of the Fourier series will be most important? Why? Consider the case where the damping is small. **(2 Points)**

3. Consider a non-relativistic charged particle moving in an arbitrary time-independent electric and magnetic field with electric potential $\Phi(\vec{r})$ and vector potential $\vec{\mathcal{A}}(\vec{r})$. The Lagrangian for this system is

$$L = T - U = \frac{1}{2} m \vec{v}^2 - q\,\Phi + \frac{q}{c}\,\vec{v} \cdot \vec{\mathcal{A}}$$

   (a) Prove that the equations of motion:

$$\vec{F} = q\left(\vec{\mathcal{E}} + \frac{\vec{v}}{c} \times \vec{\mathcal{B}}\right),$$

   where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the electric and magnetic fields, follow from the Lagrangian.

   (b) Find the Hamiltonian for this system.

   (c) Now assume that $\vec{\mathcal{E}} = \mathcal{E}_0\,\hat{i}$ and $\vec{\mathcal{B}} = \mathcal{B}_0\,\hat{k}$ are uniform, constant and perpendicular. Assuming that $\vec{r}(0) = 0$, solve for the trajectory $\vec{r}(t)$.

Statistical Mechanics

4. A certain system can be modelled as an ideal gas of point particles, but the point particles have two internal states, with energies 0 and Δ.

   (a) Show that in the canonical ensemble the partition function $Z(T,V,N)$ for the gas can be written as

   $$Z(T,V,N) = Z_0 \left(1 + e^{-\Delta/kT}\right)^N \frac{\left(VT^{3/2}\right)^N}{N!}$$

   where Z_0 is a multiplicative constant that has no effect on the equation of state. **(2 points)**

   (b) Calculate the specific heat at constant volume for the gas.

   (c) Assume further that we have *two* such gases, A and B, and that each has an internal state, but that $\Delta_A \neq \Delta_B$. Determine $Z(T,V,N_A,N_B)$, where N_A and N_B are the number of gas atoms of type A and B, respectively. **(1 point)**

   (d) Finally, if gas particles of type A can convert into type B and vice versa, calculate N_A/N_{tot} in equilibrium, where $N_{\text{tot}} = N_A + N_B$. **(5 point)**

5. A gas of N distinguishable classical non-interacting atoms is held in a neutral atom trap by a potential of the form $V(\vec{r}) = a\,r$ where $r = \sqrt{x^2 + y^2 + z^2}$. The gas is in thermal equilibrium at a temperature T.

   (a) Find the single particle partition function Z_1 for a trapped atom. Express your answer in the form $Z_1 = A\,T^{\alpha} a^{-\eta}$. Find the prefactor A and the exponents α and η. **(3 points)**

   (b) Find the entropy of the gas in terms of N, k, and $Z_1(T, a)$. Do not leave any derivatives in your answer. **(4 points)**

   (c) The gas can be cooled if the potential is lowered reversibly (by decreasing a) while no heat is allowed to be exchanged with the surroundings, $dQ = 0$. Under these conditions, find T as a function of a and the initial values T_0 and a_0. **(4 points)**

6. Consider a fictitious spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = v_0 p.$$

where $p \equiv |\vec{p}|$. We will call this particle the "offon." Assume that your offons are confined in a three dimensional sample and are non-ineracting. We will work in the Grand Canonical Ensemble.

   (a) Determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential μ (or the fugacity, $z \equiv e^{\beta\mu}$), T, and V. **(3 points)**

   (b) What is the offonic Fermi energy (μ at T =0) as a function of their density? (*Hint:* This should not involve any complicated integrals). **(3 points)**

   (c) Derive a series expansion in z for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. **(4 points)**