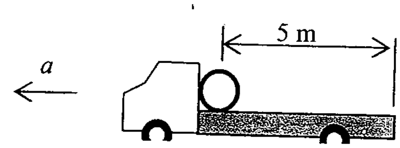
1. A section of steel pipe of radius 1 m and relatively thin wall is mounted as shown on a flat-bed truck. The driver of the truck, not realizing that the pipe has not been lashed in place, starts from rest and drives the truck forward with a constant acceleration of $0.5g$. As a result, the pipe rolls backward (relative to the truck bed) without slipping, and falls to the ground. The length of the truck bed is 5 m. (You can set $g = 10\text{m/s}^2$).



(a) (2 pt) What is the relation between the acceleration of the center of mass of the pipe and the angular acceleration of rotation of the pipe about its center of mass?

(b) (2 pt) What is the acceleration of the center of mass of the pipe as it rolls without slipping on the truckbed?

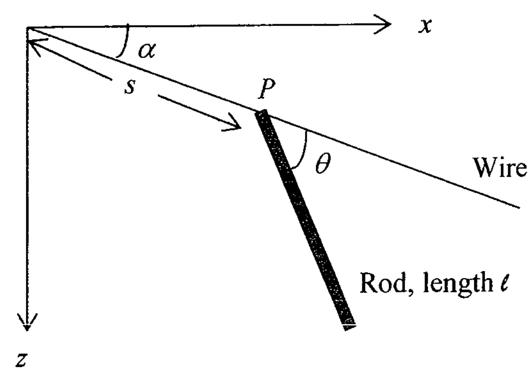
(c) (2 pt) With what horizontal velocity does the pipe strike the ground?

(d) (1 pt) What is its angular velocity at this instant?

(e) (2 pt) How far does it skid before beginning to roll without slipping, if the coefficient of friction between the pipe and ground is 0.3?

(f) (1 pt) What is its linear velocity when its motion changes to rolling without slipping?

2. A homogeneous rod of length l and weight mg slides in the vertical x-z plane along a smooth frictionless wire which is inclined at an angle α as shown. The rod can pivot about its end (point P) in the x-z plane as it slides.



(a) (2 pt) Calculate the moment of inertia of the rod about the center of mass of the rod.

(b) (3 pt) Find the Lagrangian for this system in terms of the generalized coordinates θ and s.

(c) (3 pt) Determine the equations of motion in terms of the generalized coordinates θ and s.

(d) (2 pt) From these equations, determine if pure translational motion is possible ($\theta =$ constant) and, if so, for what values of θ.

3. Consider the Lagrangian for a single particle described by the Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}, t)$.

(a) (1 pt) What is the canonical momentum $\mathbf{p}$ of the particle?

(b) (1 pt) What is the Hamiltonian of the particle? What are the natural or canonical variables of the Hamiltonian?

(c) (1 pt) What is the action for the system expressed in terms of trajectories for the particle, $\mathbf{r}(t)$, between two times t_1 and t_2.

(d) (2 pt) Consider an arbitrary infinitesimal variation in the particle trajectory, $\mathbf{r}(t) \to \mathbf{r}(t) + \delta\mathbf{r}(t)$, and require that the action W be stationary, $\delta W = 0$, provided endpoint variations vanish, $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = 0$. What is the resulting equation of motion of the particle?

(e) (1 pt) Suppose the endpoint variations are not zero, then what is δW?

(f) (1 pt) Suppose the system is invariant under a rigid coordinate translation:

$$\delta\mathbf{r} = \text{constant} : \quad \delta W = 0.$$

What do you then conclude about the momentum?

(g) (2 pt) Consider a time variation, δt_1, δt_2. Instead of changing the end times, we can change the time parameter of integration, $t \to t + \delta t(t)$, where $\delta t(t)$ is arbitrary but so chosen that $\delta t(t_{1,2}) = \delta t_{1,2}$. Then if we require δW changes only at the endpoints, $\delta W = G_2 - G_1$, where G_i depends only on dynamical variables at time t_i, $i = 1, 2$, what is the resulting equation of motion for dH/dt? (Hint: δW is stationary under interior variations of the trajectory.) Write the endpoint variation in terms of the Hamiltonian.

(h) (1 pt) If the system is translationally invariant in time, so under a rigid t translation, $\delta t = \text{constant}$, $\delta W = 0$, what do you conclude about the Hamiltonian?

4. Consider the enthalpy $H = U + pV$ of a system of N particles of mass m attached to a reservoir with which it can exchange energy and particles.

   (a) (2 pt) Use the thermodynamic identity for dH to derive the natural state variables of H.

   (b) (2 pt) Derive expressions for the conjugate variables.

   (c) (2 pt) Calculate the Maxwell relations.

   (d) (2 pt) For an ideal gas, compute H in terms of its natural variables, recalling that for an ideal gas, the entropy is

$$S = kN\left(\frac{5}{2} + \frac{3}{2}\ln 2\pi mkT - \ln N/V\right),$$

   in terms of the temperature T and the volume V.

   (e) (2 pt) Show that, in this case, the relations found in part 4b are satisfied.

5. Imagine a system of N noninteracting spinless nonrelativistic bosons of mass m in a volume V in d spatial dimensions.

   (a) (2 pt) If the possible single-particle energy levels are ε_j, what is the Bose-Einstein distribution function for the mean number of particles occupying the jth energy state, in terms of the temperature T and the chemical potential μ.

   (b) (1 pt) What is the largest possible allowed value of the chemical potential?

   (c) (1 pt) In d spatial dimensions, how many states are there in an element of phase-space $d^d p\, d^d x$?

   (d) (1 pt) Derive a formula for the number of particles N in a macroscopic box of volume V in terms of the distribution given in part 5a. Use polar coordinates in momentum space to write

$$d^d p = p^{d-1}\, dp\, A(d), \quad A(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

   where the volume of a unit sphere embedded in d dimensions, $A(d)$, is given in terms of the gamma function.

   (e) (2 pt) Show that in general as $T \to 0$ this formula cannot be satisfied, since the number of particles N is fixed, unless there is macroscopic occupation of the ground (lowest-energy) state. (This is Bose-Einstein condensation.)

   (f) (2 pt) For $d = 3$ give a formula for the relative number of bosons in the ground state, in terms of the temperature T. (Hint:

$$\int_0^\infty dx \frac{x^{a-1}}{e^x - 1} = \Gamma(a)\zeta(a),$$

   in terms of the Riemann zeta function $\zeta(a)$ defined by

$$\zeta(a) = \sum_{n=1}^{\infty} \frac{1}{n^a}.$$

   (g) (1 pt) What happens when $d = 2$?

6. Consider a "lattice gas" of N_0 distinguishable atoms in a volume V that is split up into n cells of volume b. Each cell can either be empty, contain one atom, or contain two atoms bound into a molecule with binding energy ε. If there are N_1 singly occupied cells and N_2 doubly occupied cells, the energy of the system is $E = -N_2\varepsilon$.

   (a) (2 pt) Calculate the partition function $Z_n(N_1, N_2)$.

   (b) (2 pt) Calculate the Helmholtz free energy F when all the numbers N_1, N_2, and $n - N_1 - N_2$ are large. What are the natural (canonical) variables for F?

   (c) (2 pt) Calculate the chemical potentials of the atomic and molecular species.

   (d) (2 pt) Give an expression determining the volume fraction of molecules $f_2 = N_2/n$ at temperature T and density $f_0 = N_0/n$.

   (e) (2 pt) The interstellar density of hydrogen is about one hydrogen atom per cubic centimeter, and suppose this gas is at the temperature of the cosmic microwave background 2.73 K. (This is actually not realistic.) The binding energy for the hydrogen molecule is $\varepsilon = 4.5$ eV. Use this model to predict that a large fraction of the atoms will be bound into the molecular state. Why is this expected?