Classical Mechanics and Statistical/Thermodynamics

January 2014

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\; dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

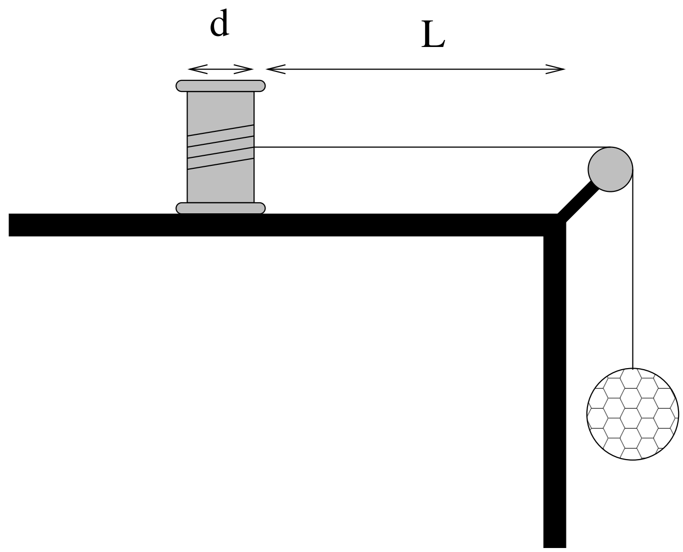
$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$
$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A
electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg
Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K
Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$
speed of light: $c = 3.00 \times 10^8$m/s

Classical Mechanics

1. A solid spool of uniform density has a mass m_s and diameter d. It rests on a frictionless table and is attached by a massless string to a hanging ball with mass m_b and radius r. The string runs over an ideal massless, frictionless pulley as shown. The system is released from rest with the spool a distance L from the edge of the table. When it is released, the spool starts to slide and rotate as it is pulled by the string. Denote the acceleration of gravity by the constant, g.



   (a) Consider the spool to be a uniform cylinder and calculate the moment of inertia of the spool rotating about its center of mass. Do not simply state the result. (1 point)

   (b) Find the constant acceleration of the spool as it moves to the right, in terms of the variables given, and g, the acceleration due to gravity. (3 points)

   (c) What is the velocity of the ball when the spool has travelled a distance L and reaches the edge of the table? (3 points)

   (d) What is the ratio of the total kinetic energy of the spool (translational and rotational) to the kinetic energy of the ball? (3 points)

2. *Wilberforce Pendulum:* A mass m is suspended from the ceiling by a long coiled spring, forming a *Wilberforce Pendulum*. The system can oscillate in the vertical direction (z) and twist about its vertical axis, (θ). The Lagrangian for the system is:

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\Omega\theta^2 - \frac{1}{2}\lambda z\theta$$

(a) Explain each term in the Lagrangian. (1 point).

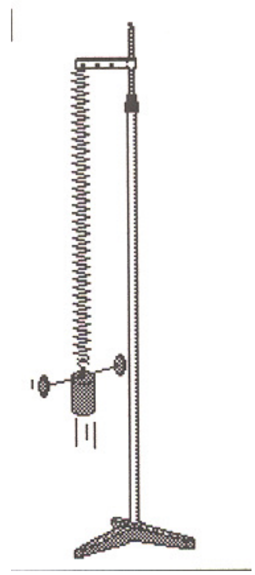
(b) Determine the equations of motion for the system. (3 points).

(c) Determine the frequencies of the normal modes. (3 points).

(d) At time $t = 0$ the ball is displaced upwards a distance z_0 from its equilibrium position, without any twist, and then released from rest. Determine the motion. You might find it helpful to define:

$$\begin{aligned} \omega_z &\equiv \sqrt{k/m} \\ \omega_\theta &\equiv \sqrt{\Omega/I} \end{aligned}$$

and/or other constants to simplify your algebra. (3 points)



3. Let $[F, G]$ denote the Poisson bracket of the quantities F and G.

   (a) Show that: (2 points)

      i. $[p_i, p_j] = 0$

      ii. $[p_i, q_j] = -\delta_{i,j}$

      where q_j is a co-ordinate and p_j is its canonical momentum.

   (b) If $\vec{L}$ is the angular momentum in three dimensions given by

   $$\vec{L} \equiv \vec{r} \times \vec{p}$$

   show that: (5 points)

      i. $[L_i, L_j] = L_k$, for i, j, k in cyclic order.

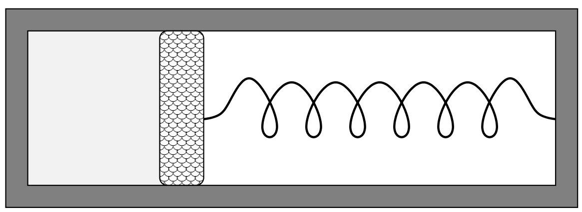
      ii. $[L^2, L_i] = 0$

   (c) Based on the above, can L_x, L_y, and L_z serve as a set of canonical momenta for some set of generalized coordinates in a central force problem? Why or why not? (3 points)

Statistical Mechanics

4. A thermodynamic system consists of n moles of an ideal mono-atomic gas confined in an insulating cylinder by a piston of cross-sectional area A. Initially the piston is locked into place so that the gas is in equilibrium with an initial volume V_0, temperature T_0, and pressure P_0. A spring of spring constant k is attached to the piston, but is initially neither stretched nor compressed. The volume occupied by the spring on the right hand side of the cylinder is a vacuum, (there is no gas to exert a pressure back on the piston.

   When the piston is released, it will compress the spring. Eventually, the system will come to equilibrium with some of the internal energy of the gas transferred to the potential energy stored in the spring which has been compressed a distance x. Denote the final volume, temperature and pressure by V_1, T_1 and P_1. You should neglect the heat capacities of the cylinder walls, the piston and the spring.



   It is trivially true that $(V_1 - V_0) = Ax$; in parts (a) and (b) you are asked to use physics to determine other relationships between x and the variables in the problem.

   (a) Find the relationship between the change in temperature of the gas, $T_0 - T_1$ and the final compression of the spring, x, in terms of the variables above. (1 point)

   (b) What is the relationship between the final pressure, P_1 and the final compression of the spring, x, in terms of the variables above? (1 point)

   (c) You are told that the when equilibrium is reached, the volume of the gas has doubled ($V_1 = 2V_0$). What will be the ratio of the final temperature to the initial temperature? Your answer should be dimensionless. (5 points)

   (d) If the above case, what will be the ratio of the final pressure to the initial pressure? Your answer should be dimensionless. (3 points).

5. Consider a system of N non-interactiong, distinguishable spin-1/2 particles in a magnetic field B at an initial temperature T. Each spin has energy $\pm g\mu_0 B/2$ depending on whether it is alligned $(-)$ or anti-alligned $(+)$ with the applied magnetic field. They have no other energy.

   (a) Show that the entropy is given by:

$$S(N, T; B) = Nk_B \ln \left(2 \cosh \frac{g\mu_0}{2k_B T}\right) - \frac{Ng\mu_0 B}{2T} \tanh \frac{g\mu_0 B}{2k_B T}.$$

   (3 points)

   (b) The magnetic field is reduced adiabatically. Show that if the field B is reduced to half its value, the temperature will also be reduced to half its value. (1 point)

   (c) Evaluate the entropy of the spin system in the limits $g\mu_0 B/k_B T \to \infty$ and $g\mu_0 B/k_B T \to 0$. Explain your answers. (1 point)

   (d) The above spin system is placed in thermal contact with an ideal mono-atomic gas of N particles in a volume V where the canonical partition function for a single gas atom is:

$$Z_{\text{atom}} = CVT^{3/2}$$

   (the value of N is the same as the number of spins). What is the entropy of the ideal gas by itself? (2 points)

   (e) Initially the two systems are in equilibrium at temperature T_0 and $g\mu_0 B/k_B T_0 >> 1$. If we adiabatically reduce the magnetic field to zero and the two systems remain in thermal contact, what is the final temperature, T_1, of the gas? (3 points)

6. Consider a non-relativistic gas of N electrons confined on a two dimensional surface of area $A = L^2$.

   (a) Show that the density of states of the gas is

   $$g(\epsilon) = \frac{mA}{\pi\hbar^2}$$

   (**3 points**)

   (b) Find the Fermi energy E_F (in terms of N and A) (**2 points**)

   (c) Find the average energy per electron at $T = 0$ (in terms of E_F) (**2 points**)

   (d) Write down the expression for the total number of particles at temperature $T \neq 0$. Use this expression to find $\mu = \mu(T)$. (**3 points**).

   (e) Take the $T \to \infty$ limit. Do you recover the classical limit? What about the $T \to 0$ limit? (**1 point**)

   (f) Calculate (in order of magnitude) E_F for a gas of density $\frac{N}{A} = 10^{16} m^{-2}$. Is this Fermi gas degenerate at room temperature and why or why not?(**1 point**)