Classical Mechanics and Statistical/Thermodynamics

January 2015

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Classical Mechanics

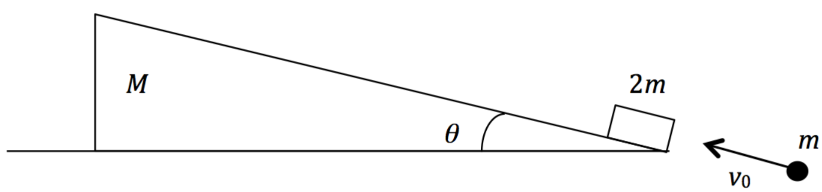
1. A block with mass $2m$ is initially at rest at the bottom edge of a wedge of mass M and angle θ. The wedge sits on a horizontal table. The table-wedge surface and block-wedge surface are frictionless. At $t = 0$ a bullet of mass m and velocity v_0 traveling parallel to the upper surface of the wedge collides with and embeds in the block.

   (a) What is the maximum height above the table reached by the block? (4 points)

   (b) At that time, what is the speed of the wedge? (2 point)

   (c) How much time does it take for the block to reach its maximum height? (2 points)

   (d) How far has the wedge moved along the table at that time? (2 points)



2. A thin rod of length ℓ and total mass m has a linear mass density (mass per unit length) given by $\lambda(x) = \alpha x$, where α is a constant. We will denote the end of the rod with vanishing mass density as $\mathcal{A}$, and the acceleration due to gravity by g.

   (a) Find α in terms of m and ℓ. (1 point)

   (b) Calculate the distance between $\mathcal{A}$ and the center of mass and express it in terms of m and ℓ. (1 point)

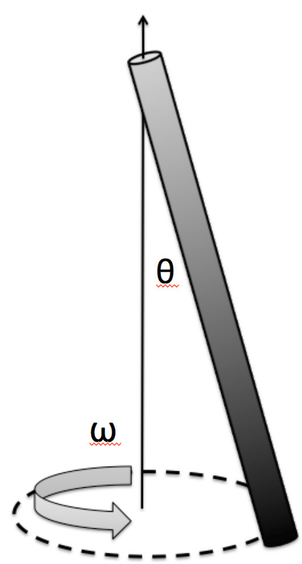
   (c) Find the moment of inertia of the rod about the end $\mathcal{A}$ about an axis perpendicular to the length of the rod and express it in terms of m and ℓ. (1 point)

   (d) The end $\mathcal{A}$ is attached to a fictionless pivot and then rotated azimuthally at constant angular frequency ω about the vertical line passing through the pivot. It is still free to rotate in the θ direction as well. Write down the Lagrangian of the system. (3 points)

   (e) Find the equation of motion for the system for θ. In the limit that $\omega^2 >> g/\ell$ there are two equilibrium solutions in which $\theta(t)$ is a constant.

      i. Show that $\theta(t) = 0$ is a solution and prove that it is unstable. (1 point).

      ii. There is a second solution in which the rod is nearly horizontal as it rotates. Derive an expression for the value of the equilibrium angle, θ_0 as a function of g, ℓ and ω and determine the frequency of small oscillations about this angle. (3 points)



3. **Angular momentum and the Rungé-Lenz[1] vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

Hint: Expand the Poisson bracket of $\vec{A}$ and use the fact that you know $\left\{H, \vec{L}\right\} = 0$. (3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this and your answer to part (c) above to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). If you use the fact that

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

your answer should not involve much algebra. (3 points)

[1]This vector is commonly called the Rungé-Lenz vector, but they were not the first to discover it. Historically it may be more accurate to call this the "Hermann-Bernoulli vector," while it is also referred to as the "Laplace-Runge-Lenz vector." Others prefer to dodge the whole question of naming rights and simply refer to it as "the axial vector."[2]

[2]The above footnote has nothing to do with the solution of this problem.

Statistical Mechanics

4. A total of n moles of a spinless, mono-atomic, ideal gas is contained in a balloon of radius r and temperature T. The balloon is an ideal black body with emissivity of unity, and it exerts a constant pressure P_0 on the gas, independent of the size of the balloon. As the balloon radiates heat, it will cool, and shrink. You should assume that otherwise the balloon is in vacuum (there is no mechanism of heat loss other than radiation, and that there is no other force on the gas other than the constant balloon pressure).

   (a) Show that the rate of temperature change for the balloon due to radiation is given by:

   $$\frac{dT}{dt} = -\frac{8\pi\sigma r^2 T^4}{5nR}$$

   where σ is the Stefan-Boltzmann constant and R is the ideal gas constant. You should assume that the balloon does not absorb any heat from its surroundings. (4 points)

   (b) As the balloon shrinks, the elastic in the balloon compresses it and therefore does work. Derive an expression for dW/dt, the rate at which the balloon does work on the gas. Express your answer in terms of the variables V, n and P_0 as well as any numerical or physical constants you need. (3 points)

   (c) Given that the initial temperature of the gas is T_0, how long does it take for the gas in the balloon to lose half of its internal energy? Assume that the gas in the balloon has a uniform temperature, and ignore the specific heat of the elastic materal of the balloon itself. (4 points)

   Note that you do not need to solve part (a) to solve parts (b) or (c).

5. Consider a lattice of N non-interacting, magnetic moments of magnitude μ, that are fixed in distinguishable locations. A magnetic field B is applied to the system such that each magnetic moment has two possible energy levels, $\pm\mu B$. Denote the number of moments in the higher energy state by the variable n.

   (a) Find an expression for the entropy in terms of N and n in the micro-canonical ensemble. (1 point)

   (b) Find the value of n for which the entropy is a maximum and sketch a graph of $S(n)$. (2 points)

   (c) Derive an expression for the energy of the system, U, in terms of N and n. In the micro-canonical ensemble this is an algebraic relationship, and not a result of ensemble averaging. (1 point)

   (d) Derive an expression for the temperature and show that it can be negative. (3 points)

   (e) If this system is put into thermal contact with a heat bath with positive temperature which way does the heat flow? Justify your answer. (3 points)

6. Consider an ideal gas of bosons confined in a two-dimensional surface of linear size L. An elementary criterion for the existence of Bose-Einstein condensation at a critical temperature T_c is

$$N_T \equiv \sum_{\epsilon_{\vec{k}} \neq 0} n(\epsilon_{\vec{k}}, \mu = 0, T = T_c) = N \tag{1}$$

where n is the Bose-Einstein distribution function and N is the total number of particles. (Assume periodic boundary conditions.)

First consider a gas which obeys the dispersion relation $\epsilon_{\vec{k}} = \hbar v k$, where v is a positive constant.

(a) Write down the Bose-Einstein condensation criterion explicitly. (1 point)

(b) Find the density of states. (2 points)

(c) Perform the sum in part a) by going to the continuum and calculate T_c in terms of N, L, v. (3 points)

(d) Now consider a gas which obeys the dispersion relation $\epsilon_{\vec{k}} = \hbar^2 k^2/2m$, where m is the particle mass. Repeat steps b) and c) and show that Bose-Einstein condensation can only happen at zero temperature. (4 points)

You may find it helpful to refer the table of formulae at the front of the exam in evaluating your sums.