Classical Mechanics and Statistical/Thermodynamics

January 2023

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}$$

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\;e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\;e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{or} \quad \log(n!) \approx n\log(n) - n$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Handy Taylor Series:

$$\log(1+x) = \sum_{n=1}^\infty (-1)^{n+1}\frac{x^n}{n}$$

$$\log(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\,\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12} \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 6.63 \times 10^{-34}$m^{2}kg/s
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\, \ell \mathrm{atm} \cdot \mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

Question 1: Consider the system shown in Fig. 1. A point particle of mass m is travelling towards a mass m_1 that is connected by a massless rigid rod of length L to another mass m_2. The velocity of mass m is initially perpendicular to the connecting rod and has magnitude v. The motion of the entire system is assumed to be confined to the 2D plane of Fig. 1. No external forces act on the system. The collision of the masses m and m_1 is assumed to be completely *inelastic*. All answers to the following questions should be expressed in terms of the masses m, m_1 and m_2 and the initial velocity v.

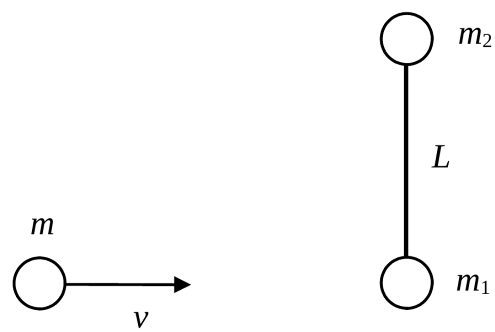

Figure 1: A point particle of mass m and initial velocity v is travelling towards a mass m_1 that is connected by a massless rigid rod of length L to another mass m_2.

(a) What is the final center-of-mass velocity of the total system after the collision? (1 point)

(b) Obtain the rotational velocity ω about the center-of-mass of the total mass/rod system after the collision. (3 points)

(c) Show that the kinetic energy decreases by,

$$\Delta \text{KE} = -\frac{m_1 m}{m_1 + m} v^2,$$

after the inelastic collision, and is thus independent of the value of m_2. (3 points)

(d) Assume instead that the collision is completely *elastic*. Find the new rotational velocity ω' of the connecting rod about its center-of-mass (including masses m_1 and m_2) after the collision. (1 point)

(e) For the elastic case, obtain two independent equations that can be used to solve for the final velocity of the mass m and the velocity of the center-of-mass of the rod after the collision. You should assume that the motion of mass m is still along the same axis as its initial motion after impacting m_1. *You do not have to solve these equations to obtain expressions for these velocities!* (2 points)

Question 2: Consider a system of three masses connected by springs, as illustrated in Fig. 2. The central mass has $m_2 = M$, while the outside masses have $m_1 = m_3 = 2M$. The springs joining the masses are each characterized by an identical spring constant k. In the following, you should assume that the motion of the masses is constrained to one dimension.

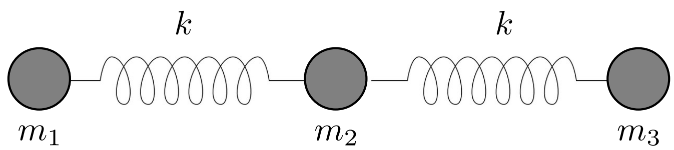

Figure 2: Three masses are connected by a pair of identical springs.

(a) Write down a Lagrangian describing the system. (1 point)

(b) Obtain equations of motion for the positions x_1, x_2 and x_3 of the three masses. (2 points)

(c) Obtain the frequencies of the normal modes describing motion of the masses near equilibrium. (3 points)

(d) Obtain the normal co-ordinates associated with the normal modes. (1 point)

(e) A periodic driving force is applied to the central mass m_2, constraining it to oscillate around its equilibrium position by a displacement $\Delta x_2 = \mathcal{A}\sin(\omega t)$ where $\mathcal{A}$ is the amplitude of the displacement and $\omega = \sqrt{k/M}$. Show that at very long times (i.e., when the system has reached a steady state) the leftmost mass m_1 oscillates out of phase with the central mass m_2 and obtain the amplitude of the displacement Δx_1 from equilibrium. (3 points)

Question 3: Consider a particle of mass m moving in three dimensions that is described by a Lagrangian,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\left(\dot{\mathbf{q}} - \Omega\mathbf{q}\right)^2$$

with generalized co-ordinates $\mathbf{q} = (q_1, q_2, q_3)$, associated generalized velocities $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \dot{q}_3)$ and Ω is a constant with dimensions 1/time.

(a) Compute the energy function associated with the Lagrangian and state whether or not it is a conserved quantity. Are the linear and angular momentum conserved? (2 points)

(b) Show that the Hamiltonian of the system is,

$$H(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \Omega\mathbf{p} \cdot \mathbf{q},$$

where $\mathbf{q}$ and $\mathbf{p}$ are the generalized position and momentum. (2 points)

(c) What does it mean for a transformation to be canonical in classical mechanics? Why is it important whether or not a transformation is canonical? (2 points)

(d) Show that the transformation,

$$\mathbf{Q} = \mathbf{q} + \frac{1}{2m\Omega}\mathbf{p},$$
$$\mathbf{P} = \mathbf{p}.$$

is canonical. Calculate the new Hamiltonian and equations of motion ($\dot{\mathbf{Q}}$,$\dot{\mathbf{P}}$) of these co-ordinates. (4 points)

Statistical Mechanics

Question 4: Consider a classical ideal gas of N molecules confined to a volume V. The system is described by the equation of state,

$$PV = Nk_BT,$$

where T is the temperature and P the pressure of the gas and k_B is the Boltzmann constant.

(a) Suppose that the heat capacity at constant volume C_V (i.e., the molecular specific heat) is known. Obtain an expression for the heat capacity at constant pressure, C_P, in terms of C_V. (3 points)

(b) For an isothermal process we have that PV is a constant. Derive the analogous expression for an adiabatic process. (3 points)

(c) Suppose that the gas under consideration is monatomic helium and it is contained in a cubic box of side length L. The box is compressed so that the side length is halved ($L \to L/2$) in an adiabatic process. Assuming that the gas remains ideal throughout the process, calculate the factor by which the pressure increases. (3 points)

(d) If the process above was repeated using nitrogen, would the pressure change be different? Why/why not? *You do not have to repeat the calculation, a conceptual explanation is sufficient.* (1 point)

Question 5: The stretching and contraction of a polymer (or analogously a rubber band) can be modelled by a chain composed of N massless segments, each of a fixed length ℓ. Each segment of the chain can be in one of two states, parallel or anti-parallel (see Fig. 3).

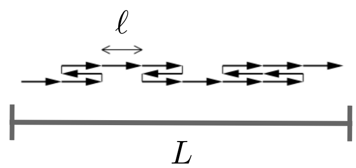

Figure 3: A polymer is modelled by a chain of N segments, each of a fixed length ℓ, that can point in one of two possible directions, parallel or anti-parallel to the overall chain. In parts (c)-(e) the chain is encased inside a narrow tube (not shown).

(a) Write an expression $\Omega(L, N)$ that corresponds to the total number of possible configurations of the chain when it has total length L (i.e., L is the end-to-end length of the chain). (2 points)

(b) Obtain an expression for the entropy $S(L, N)$ of the chain as a function of N and L. Hint: You should simplify your expression using Stirling's formula. (2 points)

Now, we assume that the polymer is placed inside a narrow tube. This containing tube is uniformly squeezed so that there is an energetic preference for the chain to be in a stretched configuration ($L \neq 0$). In this regime, an expression for the energy of the chain is,

$$E(L, N) = -\frac{\sigma L^2}{2N},$$

where σ is a constant that characterizes the applied squeezing. You should use this expression for energy for the remaining questions

(c) Show that the free energy is given by,

$$F(T, L, N) = -\frac{\sigma L^2}{2N} + \frac{k_B T}{2}\left\{ \left(N + \frac{L}{\ell}\right)\log\left(N + \frac{L}{\ell}\right) + \left(N - \frac{L}{\ell}\right)\log\left(N - \frac{L}{\ell}\right) - N\left[\log(2) + 2\log(N)\right]\right\}.$$

(1 point)

(d) Show that the tension force acting on the end points of the chain is,

$$f = -\sigma \ell x + \frac{k_B T}{2\ell}\log\left(\frac{1+x}{1-x}\right).$$

where $x = L/(N\ell)$ is the normalized chain length. Hint: The work done expanding the chain is $dW = f dL$. (3 points)

(e) Typically, a polymer under fixed tension will contract upon heating, as a result of the increasing number of possible configurations of the links in the chain for $L < N\ell$. However, as a result of the applied squeezing, there is a critical temperature below which the chain prefers to be stretched. Show that the critical temperature is given by

$$T_c = \frac{\sigma \ell^2}{k_B}.$$

Hint: Consider your expression for the tension force in part (d) when x is very small. (2 points)

Question 6: Consider a uniform two-dimensional (2D) gas of massless, ultra-relativistic spin-0 bosons confined to an area A. The gas is characterized by a total average particle number $\langle N \rangle$ and single particle energy $\epsilon = cp$, where $p = |\mathbf{p}|$ is the magnitude of the particle's momentum and c the speed of light.

(a) The Bose-Einstein distribution describes the occupation of a state with energy ϵ,

$$\langle N_\epsilon \rangle = \frac{1}{e^{\beta(\epsilon-\mu)} - 1},$$

where μ is the chemical potential and $\beta = (k_B T)^{-1}$. For the system under consideration, can the chemical potential be positive? What happens as μ approaches ϵ? (1 point)

(b) Show that the density of particles, $n = \langle N \rangle / A$, can be written in terms of the integral,

$$n = \frac{1}{2\pi\beta^2 c^2 \hbar^2} \int_0^\infty \frac{z x e^{-x}}{1 - z e^{-x}} \, dx,$$

where $x = \beta c p$ and $z = e^{\mu\beta}$. Hint: Start by writing an expression for n as an integral over all phase-space. (3 points)

(c) From the previous expression for n, show that the critical temperature for a BEC to form is,

$$T_c = \frac{2c\hbar}{k_B}\sqrt{\frac{3n}{\pi}}.$$

(3 points)

(d) A uniform non-relativistic gas of massive bosons confined to an area in 2D does not form a BEC at any temperature. What are the key differences for the relativistic gas that enable a condensate to be realized for $T < T_c$? (2 points)

(e) Use your result for the critical temperature in (c) to show that the occupation of the ground-state behaves as,

$$\langle N_0 \rangle = \langle N \rangle \left[1 - \left(\frac{T}{T_c} \right)^2 \right].$$

(1 point)