Classical Mechanics and Statistical/Thermodynamics

January 2025

12:00-3:30pm Friday, January 17, 2025.

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\, \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant $K = 8.998 \times 10^9$ N-m$^2/C^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A
electronic mass $m_e = 9.11 \times 10^{-31}$kg
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K
speed of light: $c = 3.00 \times 10^8$m/s
$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
electronic charge $e = 1.60 \times 10^{-19}$C
Density of pure water: 1.00gm/cm^3.
Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$
Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

1. A particle of mass m is fixed to move on surface defined by $z = ar^2$, where a is a positive constant. Gravity acts in the $-z$ direction.

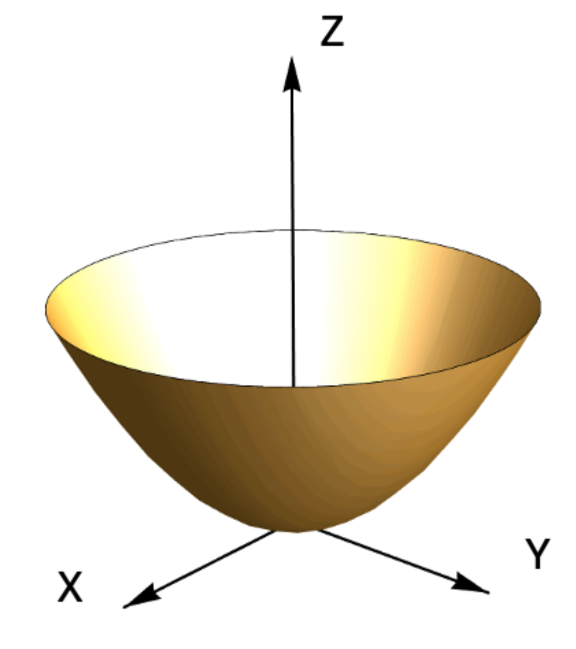

Figure 1: The particle slides without friction.

(a) Find the Lagrangian for the system in polar coordinates (r, θ). (2 points)

(b) Find the Euler-Lagrange equations for the system. (3 points)

(c) Find any independent, conserved quantities, and explain why they are conserved. (2 points)

(d) Find the canonical momenta. (1 point)

(e) Use the canonical momenta to derive the Hamiltonian. (2 points)

2. Consider an ensemble of N non-interacting electrons in a three dimensional volume V and subject to a magnetic field $\mathbf{B}$. The single-particle Hamiltonian describing such a system is,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \boldsymbol{\mu} \cdot \mathbf{B}, \tag{1}$$

where $\boldsymbol{\mu} = -(2\mu_B/\hbar)\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}$ is the spin angular momentum of the electron. In the following we will assume that the magnetic field is aligned along the z-axis.

(a) Compute an expression for the Fermi energy E_F in the absence of the magnetic field, that is $B = |\mathbf{B}| = 0$. (3 points)

(b) In the presence of a non-zero magnetic field $|\mathbf{B}| \neq 0$, compute an expression for the total number of electrons with spin-up, $N_\uparrow$, and spin-down, $N_\downarrow$, in the ground-state. (3 points)

(c) Show that the magnetic susceptibility at zero temperature and vanishing field strength ($B \to 0$) is given by,

$$\chi = \frac{\partial M}{\partial B} = \frac{3N}{2V} \frac{\mu_B^2}{E_F}, \tag{2}$$

where $M = \mu_B(N_\uparrow - N_\downarrow)/V$ is the magnetization per unit volume. (2 points)

(d) The susceptibility for classical particles diverges at low temperatures as $1/T$. Give an argument for why this is so, and why the result for electrons does not follow this behavior. (2 points)

3. A small uniform disk of mass M, radius R, and moment of inertia $I = MR^2/2$, is sitting on the flatbed of a toy electric train, both initially at rest. The disk is not attached to the train in any fashion and is sitting at the very front of the flatbed railroad car which has a length w. At time $t = 0$ the electric train is turned on and the flatbed car accelerates horizontally to the right at a constant rate of $a_0 = Ag$, where $0 < A < 1$ and g is the magnitude of the acceleration due to gravity.

   Denote the horizontal position of the center of mass of the disk with respect to the ground by $x(t)$, with $x(0) = 0$. Denote its rotation about its center of mass by $\theta(t)$, and the tangential force between the car and the disk by F_0. Assume that the acceleration due to gravity, $\vec{g}$, points down the page.

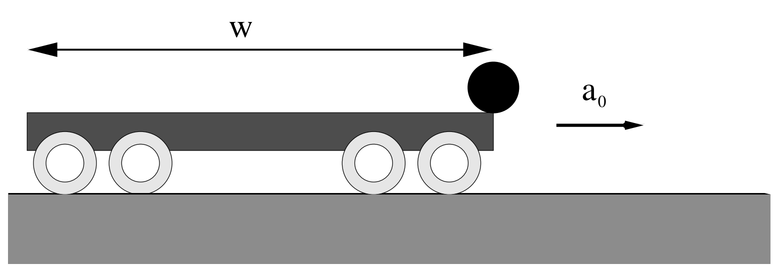

Figure 2: The disk rolls without slipping on the flatbed train car.

   (a) Write down Newton's second law for x and θ. (1 point)

   (b) Assume that the disk rolls without slipping. Find the time, t_1, that it will take for the disk to reach the back of the flatbed once it starts to accelerate. (3 points)

   (c) Find $\dot{x}(t_1)$ and $\dot{\theta}(t_1)$, the translational and rotational velocities at this time. (2 points)

   (d) Assume that at time t_1 (the instant the disk reaches the back of the train car), the train car immediately comes to a halt. If the coefficient of friction between the disk and the flatbed is μ, how much time does it take for the disk to start rolling without slipping?

   (If you do not have an answer to (c) above, write your answer in terms of the unspecified constants $v_1 \equiv \dot{x}(t_1)$ and $\omega_1 \equiv \dot{\theta}(t_1)$.) (3 points)

   (e) How far does the disk skid before beginning to roll without slipping? (1 point)

4. A particle has a position dependent mass, $m(q) = \mu_0 q^2$, where q is the generalized coordinate, and μ_0 is a constant coefficient. It is attached to a spring with spring constant $k = \gamma_0 \mu_0$, where γ_0 is also a constant. The Hamiltonian for the particle is:

$$\begin{aligned} H &= \frac{p^2}{2m(q)} + \frac{1}{2} k\, q^2 \\ &= \frac{p^2}{2\mu_0 q^2} + \frac{1}{2}\gamma_0 \mu_0\, q^2 \end{aligned}$$

(a) Derive the Hamilton equations of motion for q and p. (1 point)

(b) Show that the coordinate transformation $\{q, p\} \to \{z, p_z\}$ given by

$$z = \frac{1}{2} q^2 \qquad \text{and} \qquad p_z = \frac{p}{q}$$

is a canonical transformation. (2 points)

(c) Find a generating function $F_2(q, p_z)$ for the transformation above. (2 points)

(d) Show that the new Hamiltonian for the system after the transformation is:

$$H(z, p_z) = \frac{p_z^2}{2\mu_0} + \mu_0 \gamma_0 z$$

and derive the corresponding Hamilton equations of motion for z and p_z. Solve the above equations for the functions $z(t)$ and $p_z(t)$, given the initial values $z(0) = z_0$ and $p_z(0) = p_{z_0}$. (1 point)

(e) Consider a second transformation $\{z, p_z\} \to \{P, Q\}$ where we choose a new momentum for our description, P, where we define:

$$P \equiv \frac{p_z^2}{2\mu_0} + \mu_0 \gamma_0 z$$

That is, the new momentum is the previous Hamiltonian. Find a coordinate Q such that Q and P make up a pair of canonical coordinates. (3 points)

(f) For this transformed system, derive Hamilton's equations of motion for P and Q, and find the general solutions $Q(t)$ and $P(t)$ given the initial values Q_0 and P_0. (1 point)

5. Consider a system of N identical but distinguishable particles, each of which has a non-degenerate ground state with energy zero, and a g-fold degenerate excited state with energy $\epsilon > 0$.

   (a) Let the total energy of the system be fixed at $E = M\epsilon$, where M is the number of particles in an excited state. What is the total number of states $\Omega(E, N)$? (2 points)

   (b) What is the entropy $S(E, N)$? Assume the system is thermodynamically large. You may find it convenient to define $\nu \equiv M/N$, which is the fraction of particles in an excited state. (HINT: Use Stirlings approximation) (2 points)

   (c) Find the temperature $T(\nu)$. (1.5 points)

   (d) Invert this relation to find $\nu(T)$. (1.5 points)

   (e) Calculate the total energy of the system as a function of the temperature. (1.5 points)

   (f) Calculate the heat capacity of the system. (1.5 points)