E & M Qualifier

August 2020

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1

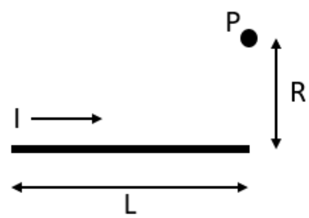
Consider a solid sphere of radius a and permittivity ϵ_0 carrying a uniform free charge density ρ.

(a) Explicitly use Gauss's Law to determine the electric field inside and outside the sphere. [3 points]

(b) The sphere is now placed in the center of a dielectric shell with permittivity ϵ. The shell's inner radius is a and outer radius is b. Determine the electric field in all three regions ($r < a$, $a < r < b$, and $b < r$). [4 points]

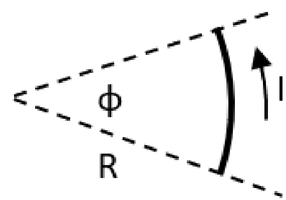
(c) Assuming the electric potential an infinite distance from the sphere is zero, determine the potential at the center of the sphere. [3 points]

Problem 2

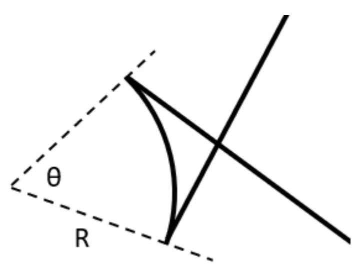


(a) A wire of length L carries a current I pointing to the right as in the figure above. Use the Biot-Savart law to determine the magnetic field at point P, which is a distance R directly above the right end of the wire. [3 points]

(b) If the wire extends infinitely to the left (but the right end is still directly below point P), what is the magnetic field at P? [3 points]



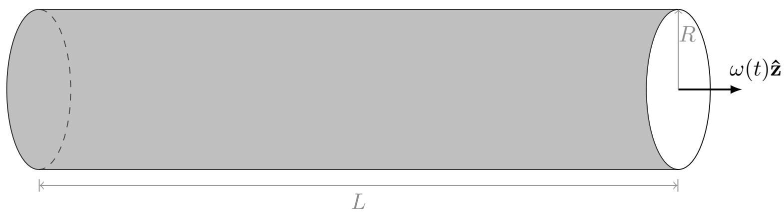
(c) Use the Biot-Savart law to find the magnetic field at the center of a circular arc with radius R and angle ϕ carrying a counterclockwise current I, as shown above. [2 points]



(d) The figure above shows two semi-infinite straight wires carrying a current I that are connected to a circular arc of angle θ, all lying in the same plane. The straight wires are tangent to the circular arc. For what angle θ is the magnetic field at the center of the arc zero? Note that while the straight wires in the drawing overlap, they are not connected except at the arc. [2 points]

Problem 3

The angular velocity $\omega(t)\hat{\mathbf{z}}$ of the cylindrical shell shown in the figure increases from zero and smoothly approaches the steady value of ω_0. The shell has an infinitesimal thickness and carries a uniform charge per unit length of $\lambda = 2\pi R\sigma$, where σ is a uniform charge per unit surface area. Assume that the shell radius $R \ll L$ and that the spin-up speed is very slow so that the displacement current can be ignored.



(a) Calculate both $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$ and $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$ everywhere during the spin-up period. [3 points]

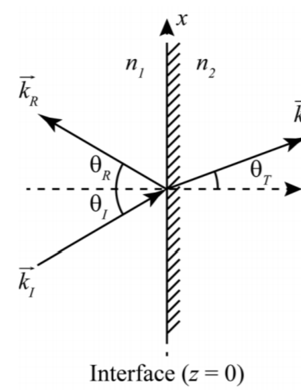
(b) Starting from the Maxwell equations what is the quantitative condition on the spin-up speed such that the displacement current can be ignored. [1.5 points]

(c) Poynting's theorem is the work-energy theorem of electrodynamics. From your knowledge of the work-energy theorem and the various energy terms in electrodynamics use qualitative arguments to arrive at the Poynting theorem; *be careful with the signs and give the theorem in integral form.* [2.5 points]

(d) The spin up is performed by an external agent that supplies power at the rate $\mathcal{P} = -\vec{\mathbf{J}} \cdot \vec{\mathbf{B}}$ per unit volume to create the magnetic field. Confirm this by evaluating Poynting's theorem over all space. [3 points]

Problem 4

Consider an interface between two dielectrics with index of refraction n_1 and n_2 (assume both materials have the same permeability), as shown in the figure to the right. A monochromatic plane wave with k-vector $\vec{k_I}$ is incident on the interface at an angle θ_I from the medium with index n_1. As a result of the interface, there is a reflected field with k-vector $\vec{k_R}$ and a transmitted field with k-vector $\vec{k_T}$.



Note that $\vec{k}$ gives the direction of propagation of the corresponding plane wave. Take the incident ($\vec{E_I}$), transmitted ($\vec{E_T}$), and reflected ($\vec{E_R}$) electric fields to all be pointing out of the page in the positive y-direction. Assume that all fields have the same frequency.

(a) Write down expressions for $\vec{E_I}$, $\vec{E_R}$, and $\vec{E_T}$ and use the boundary conditions for the electric field at the interface to derive the laws of reflection and refraction (Snell's law). That is, show that $\theta_I = \theta_R$ and $n_1 \sin\theta_I = n_2 \sin\theta_R$. [3 points]

(b) Assume now the case of normal incidence ($\theta_I = 0$). Write down expressions for the corresponding incident ($\vec{B_I}$), transmitted ($\vec{B_T}$), and reflected ($\vec{B_R}$) magnetic fields consistent with the orientation of the corresponding electric fields. [1 point]

(c) For the normal incidence case, use the boundary conditions for the electric and magnetic fields to show that the amplitude reflection and transmission coefficients are given by

$$r = \frac{E_R}{E_I} = \frac{n_1 - n_2}{n_1 + n_2} \tag{1}$$

$$t = \frac{E_T}{E_I} = \frac{2n_1}{n_1 + n_2} \tag{2}$$

respectively. [3 points]

(d) In deriving the above equations, we assumed that the reflected and transmitted fields have the same polarization as the incident field. Show that this must be the case. To do so, consider the case in which $\vec{E_I}$ is pointing in the positive y-direction and assume that the transmitted and reflected fields can have any arbitrary polarization in the $x - y$ plane. Then use the boundary conditions to show that this cannot be the case. You can again assume normal incidence ($\theta_I = 0$). [3 points]

Problem 5

A linear electric quadrupole is pictured to the right.

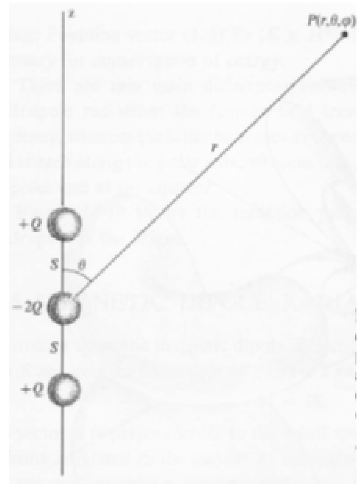
Charge flows between the middle charge and the two outer charges such that $Q = Q_0 e^{i\omega t}$, where the oscillating frequency is ω. The vector potential is

$$\vec{A}(\vec{r}, t) \approx \frac{i\mu_0}{4\pi} \frac{Q_0 s^2}{r} \frac{\omega^2}{c} e^{i(kr - \omega t)} \cos\theta \, \hat{\mathbf{z}} \tag{3}$$

and the gradient of the scalar potential is

$$\nabla V(\vec{r}, t) \approx i\omega \cos\theta A(\vec{r}, t) \, \hat{\mathbf{r}} \tag{4}$$

when the distance from the quadrupole (centered at the origin) is much larger than the wavelength ($r \gg \lambda$), which is very large compared to the quadrupole size ($\lambda \gg 2s$).



(a) Explain why the vector potential varies with the cosine of θ. Hint: Recall how the vector potential depends on the current density. [2 point]

(b) Calculate $\vec{B}(\vec{r}, t)$. Recall that $\hat{\mathbf{z}} = \cos\theta \, \hat{\mathbf{r}} - \sin\theta \, \hat{\theta}$ in spherical coordinates. [3 points]

(c) Calculate $\vec{E}(\vec{r}, t)$. [2 points]

(d) Is the quadrupole radiating electromagnetic waves out to infinity? Calculate the time-averaged Poynting vector. [3 points]

Problem 6

(a) Write down the *all space* or *boundary at infinity* solution for the electric potential as a solution to Poisson's equation $\vec{\nabla}^2\Phi(\vec{x}) = -\rho/\epsilon_0$ as an integral equation.
Hint: Think Coulumb's law. [2 points]

(b) What is the Green function $G(\vec{x}, \vec{x}')$ for this solution? [1 point]

(c) How is this solution modified in the presence of a boundary S on which the value of Φ is prescribed? [2 points]

(d) What condition must be met by the Dirichlet Green function $G_D(\vec{x}, \vec{x}')$ on S? [2 points]

(e) Write down the Dirichlet Green function for the case of a point charge located a distance $z = d$ above a grounded infinite conducting plane at $z = 0$. Show that $G_D = 0$ for $z, z' = 0$. [3 points]