# E & M Qualifier

August 18, 2022

**To insure that the your work is graded correctly you MUST:**

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

# Problem 1: Electrostatics

Consider a thin cylindrical shell of charge. The cylinder has a radius $R$ and length $L$. The axis of the cylinder lies along the $z$-axis, with one end of the cylinder at $z = 0$ and the other at $z = L$. The cylinder has a uniform charge density, with a linear density $\lambda$ along its length.

We want to determine the electric field on the axis of the cylinder.

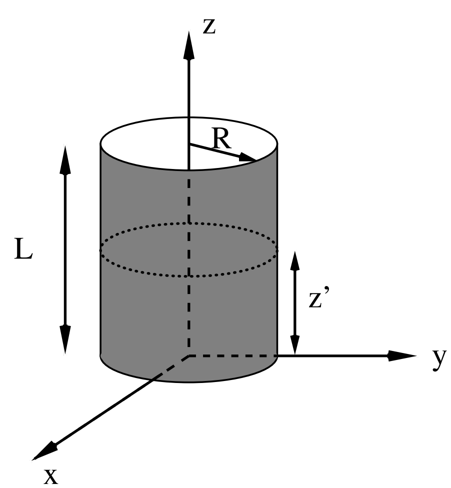
(a) Briefly explain why Gauss's Law is a bad choice for determining the electric field on the axis of this cylinder. [1 point]

(b) Sketch the expected $\vec{E}$ field along the $z$-axis. [1 point]

(c) Instead of using Gauss's Law, we can find the field by computing integrals over the charge distribution. To break it into parts, we can treat the cylinder as a collection of rings with infinitesimal height stacked on top of each other. Starting with a thin ring centered on the origin, what is the electric field on the axis of a uniform ring with radius $R$ and charge $Q$? [2 points]

(d) Use explicit integration of infinitesimal rings to compute the electric field on the $z$-axis above the cylinder described above (assume $z > L$). [3 points]

(e) Using your result from part $d$), find the electric field on the axis of the cylinder in the limits as $L/z$ and $R/z$ both approach zero. Briefly explain why that limit makes sense. (Hint: $1/(1 - x) = 1 + x + x^2 + \cdots$) [3 points]



# Problem 2: Magnetostatics

A current $I$ flows down a long straight wire of radius $R$. The wire is made of a linear homogeneous material with magnetic susceptibility $\chi_m$ and the current is distributed uniformly.

(a) What is the magnetic field $\vec{B}$ a distance $s$ from the center? (Give your answer for $s < R$ and $s > R$). [3 points]

(b) Find all the bound current densities ($\vec{J}_b$ and $\vec{K}_b$). [2 point]

(c) Calculate the net bound current $I_b$ flowing down the wire. [2 points]

(d) Verify the boundary conditions for the magnetic field $\vec{B}$. [3 points]

# Problem 3: Interaction forces

The potential distribution produced by an electric point dipole of moment $\vec{p}$ at a distance $\vec{r}$ (r = $\|\vec{r}\|$, $\vec{e_r} = \vec{r}/$r) is

$$\phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\vec{p}\cdot(\vec{\nabla}\frac{1}{r})$$

(a) Determine the electric field $\vec{E(r)}$. [2 points]

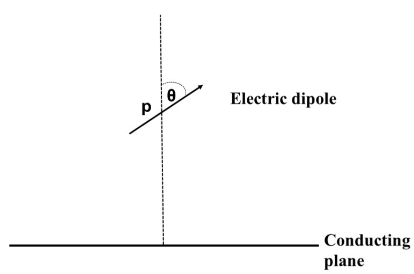
Next, the electric dipole is placed at a height $h$ above a perfectly conducting plane and makes an angle $\theta$ with respect to the normal to this plane as shown in the figure below (note: the potential from part (a) is now no longer valid).

(b) Describe the appropriate boundary conditions. [1 point]

(c) Draw the image dipole indicating the position and the orientation of the image image dipole and the direction of the force felt by the dipole. [2 points]

(d) Calculate the work required to remove the dipole to infinity. [3 points]

(e) What is the work necessary to move the dipole to infinity from a real fixed dipole in the place of the image dipole (instead of the conducting plane)? Justify your answer. [2 points]



# Problem 4: EM waves

A plane wave with electric field strength $E_0$ traveling in vacuum is normally incident on an infinite slab with permittivity $\epsilon(\omega) = 1 - \left(\frac{\omega_0}{\omega}\right)^2$.

(a) Explain what you expect will happen at frequencies above (*i.e.*, $\omega > \omega_0$) and below (*i.e.*, $\omega < \omega_0$) $\omega_0$. [1 point]

(b) Write down the expressions for all the plane waves describing the electric and magnetic fields that need to be considered and their boundary conditions at the interface for the case of $\omega > \omega_0$. [2 points]

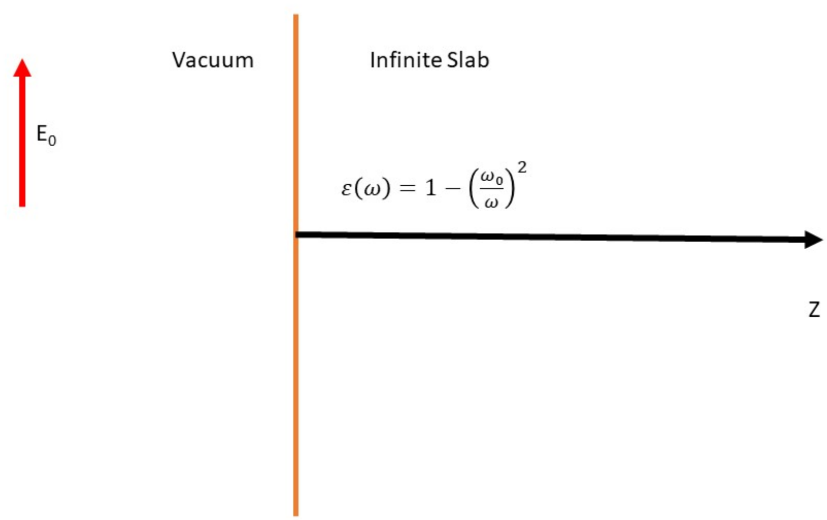
(c) Calculate the reflection coefficient when $\omega > \omega_0$. [2 points]

(d) Calculate the group velocity when $\omega \gg \omega_0$ and when $\omega \simeq \omega_0$. [1.5 points]

(e) Calculate the phase velocity when $\omega \gg \omega_0$ and when $\omega \simeq \omega_0$. [1.5 points]

(f) Calculate the skin depth for the case of $\omega < \omega_0$. [1 point]

(g) Which frequencies would you use to send information through the infinite slab? Why? [1 point]



# Problem 5: Special relativity



Observer A measures the electric and magnetic field strengths to be $\vec{E} = (\alpha, 0, 0)$ and $\vec{B} = (\alpha/c, 0, 2\alpha/c)$, respectively, where $\alpha \neq 0$. Another observer, observer B, makes the same measurements and finds $\vec{E} = (E'_x, \alpha, 0)$ and $\vec{B} = (\alpha/c, B'_y, \alpha/c)$.

(a) Write the electromagnetic field invariants [2 points]

(b) Use the field invariants to find $E'_x$ and $B'_y$ in terms of $\alpha$ and $c$. [4 points]

(c) More generally, the transformed EM fields can be written in terms of a Lorentz transformation on the field strength tensor $F^{\mu\nu}$. Write the appropriate field strength transformation law as both a tensor equation (includes indices) and as a matrix equation (without indices). [3 points]

(d) Finally, a third observer, observer C, is moving relative to observer B with constant velocity $\vec{v}$ along the positive $x$-axis of observer B. Find the electric and magnetic field strengths, $E''$ and $B''$ as observer C measures them. [1 point]

Useful Formulas:

Lorentz Boost along $x$: $\Lambda^\mu_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

EM Field Tensor: $F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -cB^z & cB^y \\ E^y & cB^z & 0 & -cB^x \\ E^z & -cB^y & cB^x & 0 \end{bmatrix}$

# Problem 6: Gauges and potentials

An essential feature of E&M is that, given time changing source fields $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$, one can determine the response of the E&M fields according to Maxwell's equations (MEs): 1. $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$, 2. $\vec{\nabla}\cdot\vec{B}=0$, 3. $\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$ and 4. $\vec{\nabla}\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}$ (in SI units).

(a) Show why introducing the potentials $\Phi(\vec{X},t)$ and $\vec{A}(\vec{x},t)$ where $\vec{B}=\vec{\nabla}\times\vec{A}$ and $\vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$ automatically solves two of the MEs. [2 points]

(b) What happens to the inhomogeneous MEs under introduction of the EM potentials? (hint: $\vec{\nabla}\times(\vec{\nabla}\times\vec{a})=\vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\vec{\nabla}^2\vec{a}$) [2 points]

(c) Show that if $\Phi$ and $\vec{A}$ are valid potentials, then so are $\Phi'=\Phi-\frac{\partial\Lambda}{\partial t}$ and $\vec{A}'=\vec{A}+\vec{\nabla}\Lambda$ for some function $\Lambda(\vec{x},t)$. [1 point]

(d) Suppose $\vec{\nabla}\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\neq 0$. One may use the freedom of choice of $\Lambda(\vec{x},t)$ to find new $\Phi'$ and $\vec{A}'$ such that $\vec{\nabla}\cdot\vec{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=0$. What is this condition called? [1 point]

(e) Using the above condition, show that the inhomogeneous MEs for the potentials decouple. What is the Green function for this equation? [2 points]

(f) Write integral equations which solve for the potentials in terms of the Green function. [1 point]

(g) How is causality exhibited by this solution? [1 point]