E & M Qualifier

August 2024

There are 5 problems. 4 of the 5 questions count as your grade on the exam. You may choose to answer only 4 questions. If you do answer all 5, the question with the lowest grade will be dropped and the remaining 4 questions will be used to grade the exam.

To ensure that your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

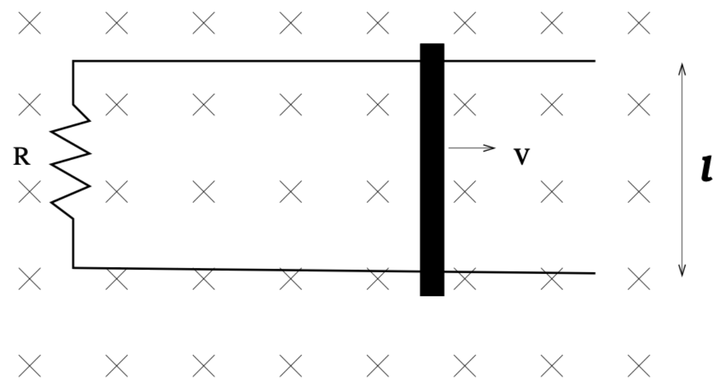
Problem 1: Electrostatics

(a) Consider an infinitely long solid cylinder of radius R. The axis of the cylinder lies along the z-axis, and the cylinder carries a charge density of $\rho = \alpha r$, where r is the distance from the axis of the cylinder. Determine the electric field inside the cylinder. [3 points]

(b) Determine the electric field outside the cylinder. [2 points]

(c) Find the potential difference between the surface of the cylinder and the axis of the cylinder. [3 points]

(d) Suppose we want to neutralize the field from this cylinder by placing it inside a thin cylindrical shell of radius $R_2 > R$. What surface charge density σ would this cylindrical shell need to have so that the electric field for $r > R_2$ is zero? [2 points]



Problem 2: Magnetostatics

A metal bar slides without friction on two parallel conducting rails a distance ℓ apart as shown, with a resistor R connecting the two rails. A constant and uniform magnetic field $\vec{B}$ points into the page everywhere.

(a) If the bar slides to the right at speed v, what current flows through the resistor? In what direction? [2 points]

(b) What is the magnetic force on the bar, and in what direction? [2 points]

(c) What is the power being dissipated in the resistor? [3 points]

(d) Calculate the mechanical power required to keep the bar moving. Comment on the conservation of energy. [3 points]

Problem 3: Dielectric Sphere

A solid sphere of radius R is made of a linear, uniform, homogeneous dielectric with dielectric constant ϵ. A surface charge has been sprayed on the sphere such that the charge density is

$$\rho\left(\vec{\mathbf{r}}\right) = \sigma \;\cos\theta \;\delta\left(r - R\right) \tag{1}$$

θ is the usual polar angle from the $+z$ axis, with $0 \leq \theta \leq \pi$

(a) Explain why the electric potential inside and outside of the sphere can be written [1 point]

$$\begin{aligned}
\Phi_i\left(r,\; \theta\right) &= \sum_\ell A_\ell \; r^\ell \; P_\ell\left(cos\theta\right), \quad r < R \\
\Phi_0\left(r,\; \theta\right) &= \sum_\ell B_\ell \; r^{-(\ell+1)} \; P_\ell\left(cos\theta\right), \quad r > R
\end{aligned} \tag{2}$$

(b) Using the boundary conditions for $\Phi\left(r,\; \theta\right)$ determine a relation between A_ℓ and B_ℓ for all values of ℓ. [2 points]

(c) Using the boundary conditions on the fields, $\vec{\mathbf{E}}$ and/or $\vec{\mathbf{D}}$, solve for the coefficients A_ℓ. [3 points]

(d) Using your results from above, calculate the functional form of the fields $\vec{\mathbf{E}}\left(r,\; \theta\right)$ and $\vec{\mathbf{D}}\left(r,\; \theta\right)$ both inside and outside the sphere. If you were not able to complete the previous questions, write equations for the fields. [2 points]

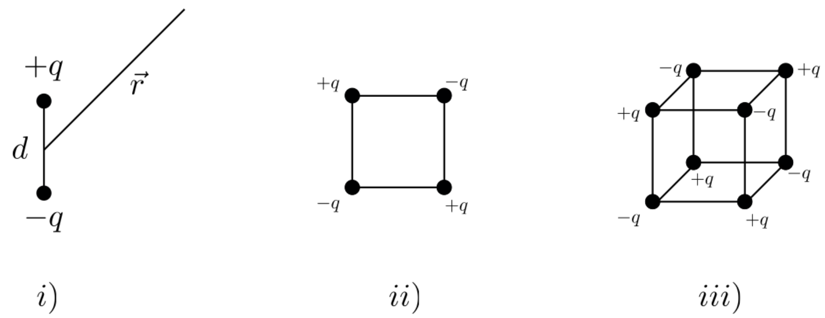
(e) Solve for the polarization, $\vec{\mathbf{P}}\left(r,\; \theta\right)$, within the sphere and surface bound charge $\sigma_b = \vec{\mathbf{P}} \cdot \hat{\mathbf{n}}$ on the sphere. Explain physically the form of the bound charge. [2 points]

In Spherical Coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi \tag{3}$$

Legendre Polynomials:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \tag{4}$$



Problem 4: Electromagnetic Multipoles

Consider the charge configurations shown in the figure.

(a) For Fig. (i), compute the exact scalar potential $V(\vec{r})$. Expand the result in $d/r \ll 1$, keeping only the leading nonzero term in the expansion. How does this potential fall off as a function of r? [2 points].

(b) Recompute your result by computing the dipole moment $\vec{p} = \int \vec{r'}\rho(\vec{r'})d\vec{r'}$ and using

$$V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

[2 points].

(c) What are the dipole moments of Fig. (ii) and Fig.(iii)? Explain. [1 point]

(d) The quadrupole moment is defined as

$$Q_{ij} = \int (3r'_i r'_j - r'^2\delta_{ij})\rho(\vec{r'})d\vec{r'}$$

and generates a potential

$$V_{\rm quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{ij}\hat{r}_i\hat{r}_j Q_{ij} \ .$$

Compute the angular dependence of $V_{\rm quad}$ for Fig. (ii) (middle figure). Assume that the square is on the $x-y$ plane, all sides are length d, and the origin is at the center. [3 points].

(e) Using symmetry arguments, or by directly computing it, determine the quadrupole moment of configuration (iii). [2 points].

Problem 5: Gauges and Potentials

(a) Write down the four Maxwell equations in a vacuum (in terms of $\vec{E}$ and $\vec{B}$ [2 points].

(b) Introduce a vector potential $\vec{A}$; what relation to $\vec{B}$ must it have to solve one of the Maxwell equation. [1 point]

(c) Introduce a scalar potential Φ. How must Φ be related to $\vec{E}$ and $\vec{A}$ in order to solve Faraday's equation? [1 point]

(d) Recast the inhomogeneous Maxwell equations in terms of Φ and $\vec{A}$. [2 points]

(e) Show the gauge freedom of the potentials Φ and $\vec{A}$ (how they transform) under introducing a scalar function $\Lambda(\vec{x}, t)$. [1 points]

(f) How can this gauge freedom be exploited to decouple the potential Φ and $\vec{A}$? [2 points]

(g) Write an integral equation which then solves the Maxwell equations for the potentials Φ and $\vec{A}$. [1 point]