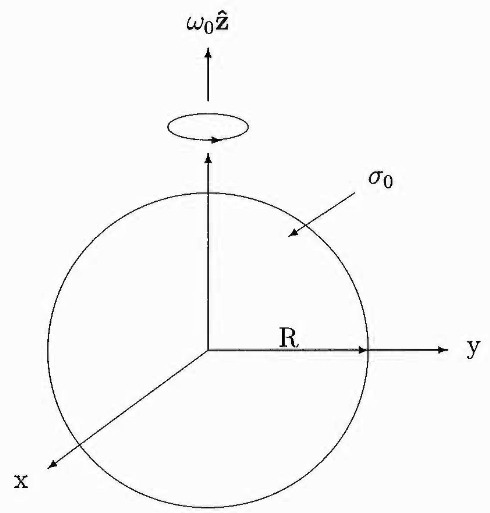


1. A rotating thin, non-conducting, sphere of radius R is covered with a uniform surface charge density σ_0. If the angular velocity is in the $+\hat{\mathbf{z}}$ direction and has a constant magnitude ω_0:

   (a) [3 pts] Compute the surface current density $\mathbf{K}(\theta, \phi)$ (magnitude and direction) as a function of R, σ_0, and ω_0.

   (b) [3 pts] Compute the magnetic dipole moment $\mathbf{m}_0$ of the rotating sphere.

   (c) [4 pts] The magnetic induction <u>exterior</u> to the sphere turns out to be a simple magnetic dipole field. Compute $\mathbf{B}(r, \theta, \phi)$ for $r > R$ assuming the vector potential is of the form:

$$\mathbf{A}(\mathbf{r})_G = \frac{\mathbf{m}_0 \times \hat{\mathbf{r}}}{r^2} \qquad \text{Gussian units,}$$

$$\mathbf{A}(\mathbf{r})_{SI} = \frac{\mu_0}{4\pi} \frac{\mathbf{m}_0 \times \hat{\mathbf{r}}}{r^2} \qquad \text{SI units.}$$

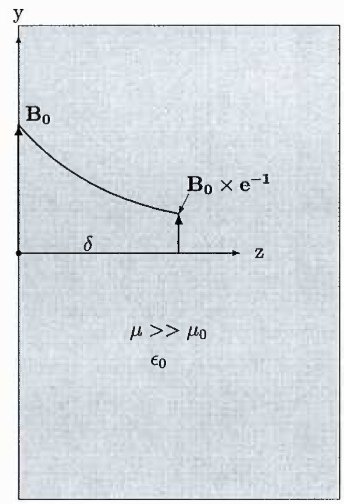


Figure 1: In Gaussian units $\epsilon_0 \to 1$ and $\mu_0 \to 1$

2. Within a transformer, oscillating magnetic fields and their associated electric fields penetrate into the transformer's iron core producing "eddy" currents which heat and frequently destroy the transformer. In this problem you are to analyze the depths to which these currents penetrate and the phase difference between the driving harmonic **B** field and the lagging eddy current.

   A large slab of permeable ($\mu >> \mu_0$) conductor with conductivity $\sigma > 0$ and with negligible permittivity ($\epsilon = \epsilon_0$) is located in the x-y plane at $z \geq 0$ as shown in the figure. A low frequency wave, $\omega << \sigma/\epsilon_0$, whose magnetic induction is the real part of

$$\mathbf{B} = B_0 e^{i(kz-\omega t)}\hat{\mathbf{y}},$$

   diminishes as z increases because k is complex.

   (a) [2 pts] Give Maxwell's 4 macroscopic equations appropriate for this material ($\rho = 0, \mathbf{D} = \epsilon_0\mathbf{E}, \mathbf{B} = \mu\mathbf{H}$, and $\mathbf{J} = \sigma\mathbf{E}$).

   (b) [2 pts] Use Maxwell's equations to find the complex wave number k as a function of ω, σ, μ, and ϵ_0.

   (c) [3 pts] The depth at which the amplitude reaches e^{-1} times its original value is called the skin depth, δ. The skin depth diminishes with the wave's frequency. For low frequency waves ($\omega << \sigma/\epsilon_0$) determine δ.

   (d) [3 pts] For low frequency waves compute the phase lag of the eddy current density $\mathbf{J}$ relative to the magnetic induction $\mathbf{B}$.

3. (a) [1 pts] Give the 4-current $J^\alpha(x^\beta)$ for the static surface charge density σ_0 shown in the figure (a thin uniform and infinite sheet of charge located at $z = 0$ in the lab).

(b) [2 pts] Give the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ caused by the static surface charge.

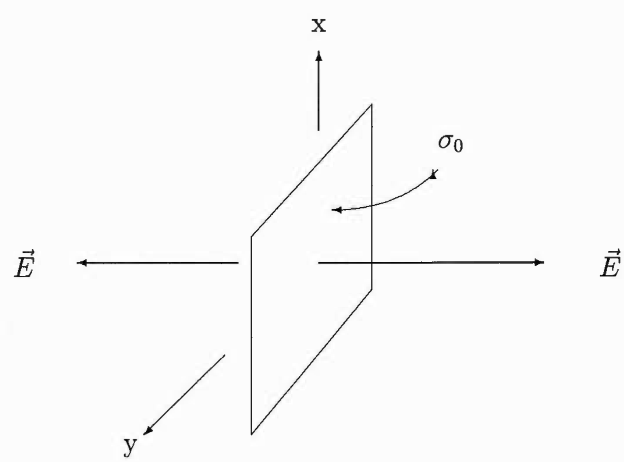
(c) [2 pts] Compute the 4-current $J'^\alpha(x'^\beta)$ in a frame that moves with speed $v < c$ in the positive x-direction relative to the lab (v isn't necessarily small).

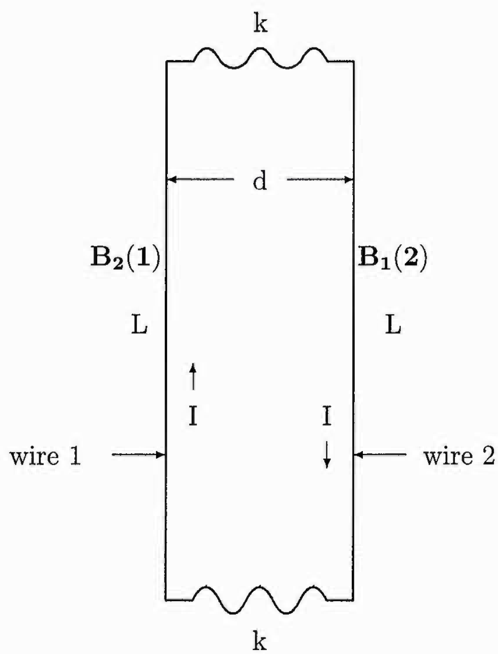
(d) [1 pts] What is the surface charge density σ' in the moving frame?

(e) [1 pts] What is the electric field $\mathbf{E}'$ in the moving frame?

(f) [1 pts] What is the surface current density $\mathbf{K}'$ in the moving frame?

(g) [2 pts] What is the magnetic induction $\mathbf{B}'$ in the moving frame?





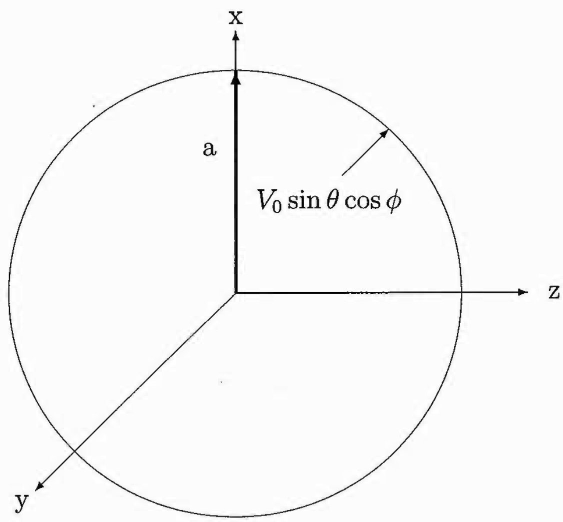
4. A current balance consists of two very long rigid parallel wires of lengths L that are connected at each end by springs (see the figure). The spring constant of both springs is k and the equilibrium distance between the wires, d(I), depends on the current. Assume $d << L$.

   (a) [2 pts] If a current I flows through the closed circuit of the 2 wires and 2 springs, find an expression for the magnetic induction $B_1(2)$ created by the first wire at the location of the second. What is the direction of this $\mathbf{B}_1(2)$ field (give the direction as up, down, left, right, into, or out of the page)?

   (b) [2 pts] Find an expression for the magnetic force $\mathbf{F}_1(2)$ on the second wire due to the $\mathbf{B}_1(2)$. What is the direction of this force?

   (c) [2 pts] Find an expression for the magnetic force $\mathbf{F}_2(1)$ on the first wire due to the magnetic induction created by the second. What is the direction of this force?

   (d) [4 pts] Are the springs stretched or compressed from equilibrium? Using the above results, find an expression for the current as a function of the amount the springs are stretch/compressed.



5. In spherical polar coordinates the solution to the Laplace equation $\nabla^2\Phi(r,\theta,\phi) = 0$, for a spherical region $r_1 < r < r_2$ can be expanded in terms of spherical harmonics in the following form:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\ell=\infty} \left(A_{\ell,m}\, r^{\ell} + \frac{B_{\ell,m}}{r^{\ell+1}} \right) Y_\ell^m(\theta,\phi),$$

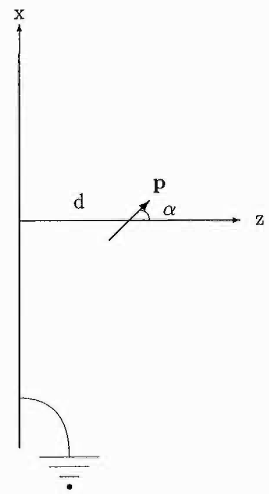
where $A_{\ell,m}$ and $B_{\ell,m}$ are constants.

(a) [3 pts] If the potential $\Phi(r,\theta,\phi)$ is given on a sphere $r = a$ but satisfies the laplace equation everywhere else, what is the form of the potential inside $(0 \le r < a)$ the sphere? Outside $(a < r < \infty)$ the sphere?

(b) [7 pts] For the particular potential given in the figure, $\Phi(r = a,\theta,\phi) = V_0 \sin\theta\cos\phi$, what is the potential inside the sphere? Outside $(a < r < \infty)$ the sphere?

Recall that the spherical harmonics are ortho-normal on the sphere and for $\ell = 1$

$$\begin{aligned} Y_1^{-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, \\ Y_1^{0}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\ Y_1^{1}(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}. \end{aligned}$$



6. A point dipole with dipole moment $\mathbf{p} = p(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})$ is located on the z-axis a distance d from a large flat grounded conducting plate situated in the z=0 plane (see the figure).

   (a) [2 pts] The part of the total potential in the region $z > 0$ caused by the induced surface charge on the grounded conductor at $z = 0$ is the same as the potential of an image dipole. What is the dipole moment $\mathbf{p}_i$ of the image dipole and where is it located?

   (b) [3 pts] What is the total electrostatic potential in the region $z \geq 0$?

   (c) [3 pts] How much work must be done to remove the dipole from $z = d$ to $z = +\infty$?

   (d) [2 pts] When at $z = d$ what force does the dipole experience?

   Hint: The electrostatic potential caused by an ideal point dipole located at the origin ($\mathbf{r} = 0$) with dipole moment $\mathbf{p} = p^x\hat{\mathbf{x}} + p^y\hat{\mathbf{y}} + p^z\hat{\mathbf{z}}$ is

$$\Phi_G(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad \text{Gaussian units}$$

$$\Phi_{SI}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad \text{SI units}$$