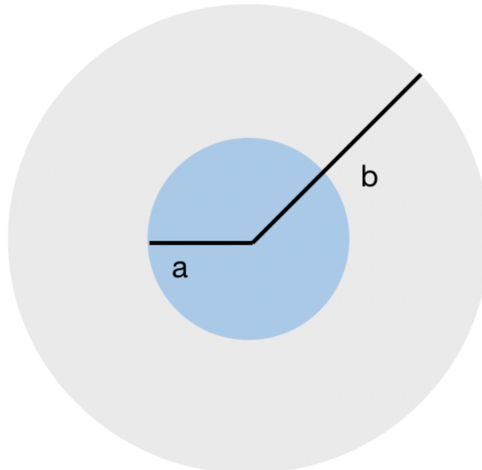
E & M Qualifier

January 2023

To ensure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
    (a) the first number is the problem number,
    (b) the second number is the page number for **that** problem (start each problem with page number 1),
    (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.



Problem 1: Electrostatics

Consider a solid sphere of radius a that carries a charge density that is a function of distance from the center

$$\rho(r) = \rho_0 \, \frac{r}{a} \; .$$

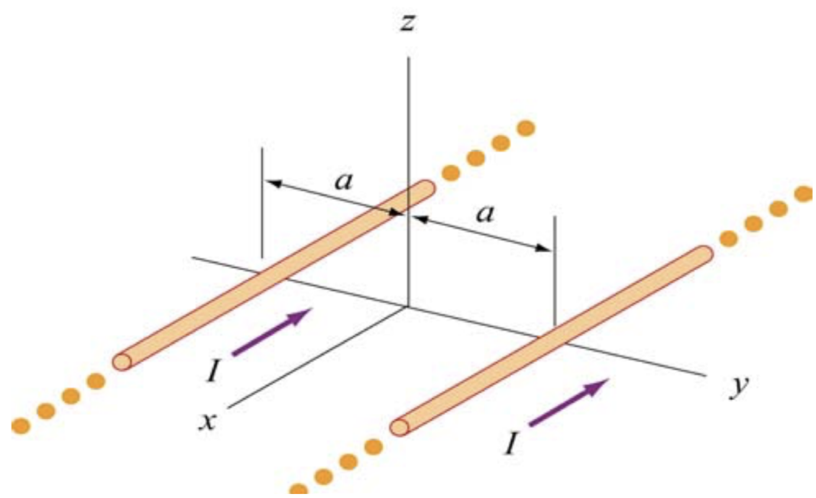
(a) Determine the electric field everywhere inside and outside the sphere. [2 points]

(b) Assuming that the electric potential very far from the sphere is zero, find the electric potential at the center of the sphere. [2 points]

(c) We now place this sphere inside a dielectric shell that has inner radius a, outer radius b, and electric susceptibility $\chi > 0$, as shown in the figure. The dielectric carries no net charge. Without doing any calculations, explain in words how you expect the potential at the center of the original sphere to change. Will it increase or decrease? Why? [2 points]

(d) Determine the new potential at the center of the sphere. [3 points]

(e) Determine the bound charge density ρ_b and the bound surface charge density σ_b on that dielectric shell. [1 point]



Problem 2: Magnetostatics

Two infinite wires carry currents I in the same direction (negative x-axis), as shown in the figure. The wires are separated by a distance $2a$. Call the wire that intersects the y-axis at $(0, -a, 0)$ "wire 1" and the wire that intersects the y-axis at $(0, a, 0)$ "wire 2".

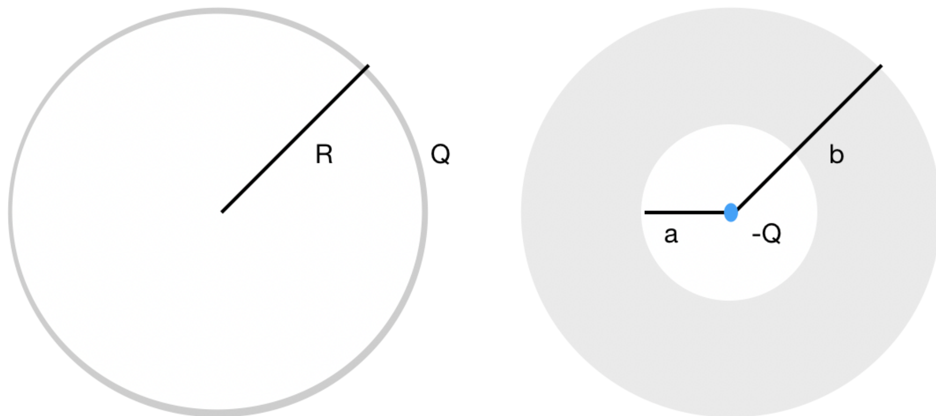
(a) Sketch the magnetic field lines produced by the two wires on the y versus z plane. Then, determine the force(s) exerted per unit length on wire 2 by wire 1, and on wire 1 by wire 2. Are the forces attractive or repulsive? [3 points]

(b) Calculate the direction and magnitude of the magnetic field produced by each wire at an arbitrary point $(0, 0, z)$. Then, calculate the total magnetic field. [2 points]

(c) Determine the value of z at which the field becomes maximum and write down the maximum value of the field. [2 points]

In the remainder of the problem, the wires are no longer parallel, but *orthogonal* to each other.

(d) Place wire 1 along the x-axis, with its current going in the positive x-direction. Place wire 2 along the y-axis, with its current going in the positive y-direction. The two wires come very close to each other near the origin, but do not actually intersect. Determine the magnitude and direction of the magnetic field at the points $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. [3 points]



Problem 3: Electromagnetic Energy

Consider an infinitesimally-thin spherical shell of radius R with a charge Q that is uniformly distributed throughout the shell. This corresponds to the figure on the left panel.

(a) Express the scalar potential as a function of the distance r from the center of the shell. Assume the potential is zero at an infinite distance from the center of the sphere. [2 points]

(b) How much work would it take to assemble the shell from charges at infinity [2 points].

Now consider instead a point charge $-Q$ that is at the center of a charge-neutral spherical conducting shell of inner radius a and outer radius b. This corresponds to the figure on the right panel. Note that the conducting shell is not infinitesimally thin.

(c) What is the charge on the inner surface of the shell? What is the charge on the outer surface of the shell? [1 point]

(d) How much work would it take to assemble the point charge and conducting shell from charges at infinity? [3 points]

(e) How much work would it take to move the point charge out to infinity (through a tiny hole drilled in the shell)? [2 points]

Problem 4: Standing Waves

(a) Consider an electromagnetic standing wave with electric field given by

$$\mathbf{E}(z,t) = E_0 \sin(kz)\cos(\omega t)\,\hat{\mathbf{i}}$$

Prove that it is a superposition of two plane waves. For each wave, write down the full form of the electric field. State the direction of propagation of the plane wave, and the polarization. [2 points]

(b) Determine the full form of the magnetic field associated with the standing wave. *Hint:* first find the magnetic fields corresponding to the two plane waves. [1 points]

(c) Determine the Poynting vector of the standing wave. [1 points]

(d) Compute the total amount of energy flowing in the time interval $t = 0$ to $t = \pi/(2\omega)$ through a surface $A\hat{\mathbf{k}}$, where A is the area of the surface and $\hat{\mathbf{k}}$ is the outward normal direction of the surface. Take the surface to be located at $z = \pi/(4k)$. [2 points]

(e) Compute the total amount of energy flowing through the same surface, but in the time interval $t = \pi/(2\omega)$ to $t = \pi/\omega$. From here, discuss the flow of electromagnetic energy through the surface in the full interval $t = 0$ to $t = \pi/\omega$. Take the surface to be located at $z = \pi/(4k)$. [2 points]

(f) Suppose the surface were made of a material that completely absorbs incident radiation. Compute the total radiation pressure on the surface between $t = 0$ to $t = \pi/(2\omega)$, between $t = \pi/(2\omega)$ to $t = \pi/\omega$, and for the whole interval $t = 0$ to $t = \pi/\omega$. Take the surface to be located at $z = \pi/(4k)$. [2 points]

Possibly useful identities:

$$\begin{aligned}
\sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \\
\sin(\theta_1 - \theta_2) &= \sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2)\ .
\end{aligned}$$

Problem 5: Electromagnetic Waves

This problem is about the Drude model of charge transport in metals, which assumes that electrons experience a velocity-dependent damping force described by a phenomenological constant γ.

(a) Write down Maxwell's Equations for a metal with conductivity σ. As usual, the permittivity ϵ and permeability μ are given by $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$. The current is given by $\mathbf{J} = \sigma\mathbf{E}$. Taking plane wave solutions of the form $\mathbf{B} = \mathbf{B_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ and $\mathbf{E} = \mathbf{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, derive the dispersion relation $\omega(k)$. Your answer should involve σ, ϵ, and μ. (You may consider $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ for this part of the problem). [3 points]

(b) In the Drude model, the AC conductivity is described by:

$$\sigma(\omega) = \frac{Ne^2}{m_e(\gamma - i\omega)}$$

Here, N is the total number of electrons and m_e is the mass of an electron. Consider a conducting medium where $\gamma \ll \omega$. Show that the conductivity is

$$\sigma(\omega) = i\varepsilon_0 \frac{\omega_P^2}{\omega} + \mathcal{O}(\gamma/\omega)$$

where $\omega_P^2 = \frac{Ne^2}{\varepsilon_0 m_e}$ is a constant. [1 point]

(c) Using your results from parts (a) and (b), derive the real and imaginary part of the refractive index at frequencies $\omega < \omega_P$. [3 points]

(d) Find the penetration depth of the wave, as a function of frequency for frequencies $\omega < \omega_P$. [3 points]

Problem 6: Poynting Vector

A plane wave of angular frequency ω exists in an isotropic homogeneous medium with electric permittivity ϵ and magnetic permeability μ. Both ϵ and μ are real and positive. The notation for the relevant vectors is the following: $\mathbf{E}(\mathbf{r}, t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, $\mathbf{H}(\mathbf{r}, t) = \mathbf{H_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$. Assume that the amplitude $\mathbf{E_0}$ is real and positive. Bold font implies vectors; the same symbol without the bold font implies the magnitude of the relevant vector.

(a) Solve Maxwell's Equations to prove that $\mathbf{k} = \omega\sqrt{\epsilon\mu}\,\hat{\mathbf{z}}$. You can assume that $\mathbf{E_0}$ is along the x-axis. [3 points]

(b) Show that $\mathbf{H_0} = E_0\sqrt{\epsilon/\mu}\,\hat{\mathbf{y}}$. [1 point]

(c) Write down the solution for the magnitude and direction of the Poynting vector $\mathbf{S}(\mathbf{r}, t)$. Your solution should only involve the quantities ω, μ, ϵ, and E_0. Determine the time average of the Poynting vector. [2 points]

(d) Consider the case where $\epsilon < 0$, and $\mu > 0$, with both being real. Assume that E_0 is real. Write down solutions for $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{H}$. Find the Poynting vector $\mathbf{S}(\mathbf{r}, t)$ and its time average. Does the wave propagate? [2 points]

(e) Consider the case where both $\epsilon < 0$, and $\mu < 0$. Assume that E_0 is real. Write down solutions for $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{H}$. Find the Poynting vector $\mathbf{S}(\mathbf{r}, t)$ and its time average. Does the wave propagate? [2 points]