Quantum Mechanics
Qualifying Exam - Fall 2020

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Harmonic oscillator wave functions

$u_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$

$u_1(x) = (\frac{m\omega}{\pi\hbar})^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega x^2}{2\hbar}}$

Problem 1: Time Dependent Operators (10 pts)

Consider a quantum system described by two basis states, which are eigenvectors of an operator with eigenvalues "up" and "down" respectively:

$$|\psi_{up}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\psi_{down}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

The system evolves in time, with a time evolution operator given by

$$\hat{U}(t+\Delta t, t) \equiv \hat{U}(\Delta t) = \begin{pmatrix}\cos\nu\Delta t/2 & -\sin\nu\Delta t/2\\ \sin\nu\Delta t/2 & \cos\nu\Delta t/2\end{pmatrix} \tag{1}$$

where ν is some parameter and t denotes the time. The system begins at time $t = 0$ in the state $|\psi_{up}\rangle$.

(a) [1 pt] Find the time T when the state evolves and first becomes pure $|down\rangle$. Assume that you didn't observe the system as it evolved.

(b) [3 pt] Now let's do some cases where you *do* observe the qubit as it evolves. Suppose that the first time you measure it is at time $T/2$, and the second time you observe it is at time T, where T is the time you found in part (a). When you observe the system, you're getting the eigenvalues *up* or *down*. The system started at time $t = 0$ in the state $|\psi(0)\rangle = |up\rangle$.

Find the probability that the two measurements you make are (i) *up* and *up*, (ii) *up* and *down*, (iii) *down* and *up* and (iv) *down* and *down* and fill out the following table:

Measured value at $T/2$	Measured value at T	Joint Probability
up	up	?
up	down	?
down	up	?
down	down	?

(c) [3 pt] Repeat part (b), but assuming that the initial state at time $t = 0$ is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|up\rangle + |down\rangle) \ . \tag{2}$$

(d) [3 pt] Go back to part (b) and assume that at time $t = 0$ the system is in the state $|\psi(0)\rangle = |up\rangle$. Now you measure it N times at regular intervals: $t_1 = \frac{T}{N}, t_2 = \frac{2T}{N}, t_3 = \frac{3T}{N}, \ldots, t_N = T$. Determine the probability that all N measurements yield the result "up". Take the limit of fixed T but large N (with $N \gg 1$). What does your answer reduce to?

Problem 2: Harmonic Oscillator (10 pts)

The quantum harmonic oscillator Hamiltonian can be written

$$\hat{H} = \hbar\omega_o(\hat{a}^\dagger\hat{a} + \frac{1}{2})$$

where we can write $\hat{a}$ and $\hat{a}^\dagger$ in terms of the dimensionless position $\hat{x}$ and momentum $\hat{p}$ operators:

$$\hat{a} = (\hat{x} + i\hat{p})$$
$$\hat{a}^\dagger = (\hat{x} - i\hat{p})$$

so that $[\hat{a}, \hat{a}^\dagger]$=1. The operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

The energy eigenstates of the system can be denoted by $|n\rangle$, and $\hat{H} = \hbar\omega_o(n + \frac{1}{2})|n\rangle$.

Consider a particle in a 1D harmonic potential so that it can be treated as an ideal quantum harmonic oscillator. At t=0 the particle is prepared in the initial state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(\ |0\rangle + |1\rangle\ ). \tag{3}$$

(a) [2 pts] Write down the explicit time dependent expression for the ket $|\psi(t)\rangle$ in this basis.

(b) [2 pts] Find $\langle x(0)\rangle$, $\langle p(0)\rangle$,

(c) [2 pts] Find $\langle x(t)\rangle$, and $\langle p(t)\rangle$.

(d) [2 pts] Use Ehrenfest's Theorem to derive expressions for $\frac{d}{dt}\langle x(t)\rangle$ and $\frac{d}{dt}\langle p(t)\rangle$.

(e) [2 pts] Use your expressions above to solve for $\langle x(t)\rangle$ and $\langle p(t)\rangle$.

Problem 3: 1-D potentials (10 pts)

a) First, consider a particle (of mass m) in a 1D infinite well potential,

$$V(x) = \begin{cases} 0, & 0 < \text{x} < \text{L} \\ \infty, & \text{otherwise} \end{cases}$$

It is initially in the ground state $|\Psi(t=0)\rangle = |\psi_1\rangle$, where $|\psi_i\rangle$ are the energy eigenstates.

(i) [3 pts] What is the uncertainty on the particle's momentum?

(ii) [0.5 pts] Does the particle's probability density (in position space) depend on time? Explain your answer.

(iii) [0.5 pts] The position of the particle is measured and found to be $L/2$. Sketch or describe the wave function immediately after the position measurement.

(iv) [1 pt] The energy of the particle is measured immediately after the position measurement. What are the possible outcomes of the energy measurement? Explain your answer.

(v) [0.5 pts] The position of the particle is re-measured immediately after the energy measurement. What are the possible outcomes of the measurement? Explain your answer.

(b) [0.5 pts] A particle (of mass m) in a 1D infinite well potential (extending from $x = 0$ to $x = L$) is initially in the state $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$, where $|\psi_i\rangle$ are the energy eigenstates. Does the particle's probability density (in position space) depend on time? Explain your answer.

(c) [4 pts] Now, consider a particle in the following potential:

$$U(x) = \begin{cases} \infty \text{ for } x < 0 \\ -V_0 a \delta(x-a) \text{ for } x > 0 \end{cases} \tag{4}$$

where V_0 and a are positive constants. Derive a transcendental equation which could be solved to yield the allowed wave number k of the bound state.

Problem 4: Angular momentum (10 pts)

An electron is in the $\ell = 1$ state of the hydrogen atom. A magnetic field is applied in the $\hat{n}$ direction, with the Hamiltonian

$$\mathcal{H} = \alpha B_0 \hat{n} \cdot \mathbf{L}.$$

Ignore spin effects.

(a) [4 pts] Write down the operator $\hat{n}\cdot\mathbf{L}$ matrix, with $\hat{n} \equiv (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ a unit vector in spherical coordinates (θ, ϕ) What are the eigenvalues of $\hat{n}\cdot\mathbf{L}$? Compare them with the eigenvalues of L_z and interpret your result.

(b) [3 pts] Assume now that $\hat{n}$ is restricted to the $x - y$ plane. Compute the eigenvectors of $\hat{n} \cdot \mathbf{L}$.

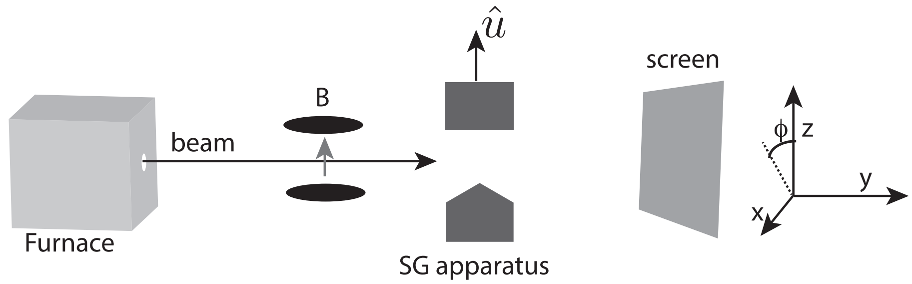
(c) [3 pts] Using your result in b), if the state is initially prepared to be at the $|\ell = 1, m = 0\rangle$ state, calculate the probability of finding it in the $|\ell = 1, m = 1\rangle$ state at time t.

Hint:

$$L_{\pm}|\ell, m\rangle = \hbar\sqrt{(\ell \mp m)(\ell \pm m + 1)}|\ell, m \pm 1\rangle$$
$$\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$$

Problem 5: Spin 1/2 particle dynamics (10 pts)

Consider an unpolarized beam of spin 1/2 silver atoms emerging along $\hat{y}$ from a furnance, through a small piercing on one side. The beam passes through a Stern-Gerlach apparatus before striking and depositing on a screen (see Figure).



(a) [2 pts] Setting $|\vec{B}| = 0$ for now, describe the distribution of the silver atoms on the screen if $\hat{u}$ is aligned with $\hat{z}$. Now consider that $\hat{u}(t)$ rotates in time about $\hat{y}$ and $\hat{u}(t) \cdot \hat{y} = 0$. Describe the distribution in this second case.

(b) [8 pts] Imagine now that the beam emerges from the furnace polarized in $\hat{z}$ with a magnetic moment of $\mu = \mu_0 s$ and an eigenvalue of $s_z = +1/2$. You apply a magnetic field B along $\hat{x}$ with the apparatus configured such that the passing atoms experience this field for exactly a duration τ. If the SG apparatus is oriented along the z axis, derive the probability of finding $s_z = -1/2$ as a function of B.

Problem 6: Perturbation theory (10 pts)

Consider a non-relativistic particle of mass m moving in a three dimensional potential given by:

$$V(x) = \frac{1}{2}k(x^2 + y^2 + z^2)$$

a) What is the ground state energy and first excited state energy for this potential (1 points)

Now there is a perturbation applied so the potential becomes

$$V(x) = \frac{1}{2}k(x^2 + y^2 + z^2 + \lambda xy)$$

where λ is a small parameter.

b) Calculate the ground state energy to first order in λ. (1 points)

c) Calculate the ground state energy to second order in λ. (4 points)

d) Calculate the first excited state energies to first order in λ. (4 points)

Hint: The coordinates x and y can be expressed in terms of the appropriate raising and lowering operators for the unperturbed problem.