Quantum Mechanics
Qualifying Exam - Fall 2023

Notes and Instructions

- *There are 5 problems but only 4 problems will count to your grade.* If you chose to solve all 5, the problem you score the least will be discarded. Attempt at least four problems as partial credit will be given.
- Write your alias on the top of every page of your solutions. *Do not write your name.*
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3).
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\,, \qquad p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta \mathbf{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_2^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: 1D quantum well

Consider a particle of mass m in a 1D infinite square well of width L:

$$V(x) = 0, \quad |x| \leq \frac{L}{2} \qquad V(x) = \infty, |x| > \frac{L}{2}. \tag{1}$$

a) Show that the states:

$$\begin{aligned} \psi_n(x) &= \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L} x\right) \quad , \quad n = 1, 3, 5, ... \\ \psi_n(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \quad , \quad n = 2, 4, 6, ... \end{aligned} \tag{2}$$

are the normalized energy eigenstates for this well, and give the energy eigenvalues for these states. (1.5 points)

b) Consider the (normalized) state:

$$\Phi(x) = \sqrt{\frac{30}{L^5}} \left(\frac{L^2}{4} - x^2\right) \tag{3}$$

For a particle in the state $\Phi(x)$, the expectation values $\langle x \rangle = 0$ and $\langle p \rangle = 0$. Show and/or explain briefly these expectation values. (1.5 points)

c) For a particle in the state $\Phi(x)$, calculate the expectation value of the energy, $\langle H \rangle$. Explain why your result agrees with your answers to part a) above. Hint: $\pi^2/2 \approx 4.935$. (3 points)

d) For a particle in the state $\Phi(x)$, calculate the probability of measuring the energy to be E_1, the ground-state energy of the well. Show your work (there are calculators available to determine the numerical result). (2 points)

e) Explain how your answers to parts c) and d) agree with each other, or if you didn't answer part d) use your result for part c) to predict an approximate answer to part d). (2 points)

You might find the following integrals useful:

$$\begin{aligned} \int \sin^2(ax)dx &= \frac{1}{2}\left(x - \frac{1}{2a}\sin(2ax)\right) \\ \int \cos^2(ax)dx &= \frac{1}{2}\left(x + \frac{1}{2a}\sin(2ax)\right) \\ \int x^2 \sin(ax)dx &= \frac{2x}{a^2}\sin(ax) - \frac{a^2x^2 - 2}{a^3}\cos(ax) \\ \int x^2 \cos(ax)dx &= \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2 - 2}{a^3}\sin(ax) \end{aligned}$$

PROBLEM 2: Spin $1/2$ system

In the z basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, a muon (spin 1/2) is in the state

$$\chi = A \begin{pmatrix} i \\ \sqrt{3} \end{pmatrix}.$$

a) Determine the normalization constant A. (1 point)

b) What are the probabilities to find the muon pointed in the positive x, y and z directions? (2 points)

c) Suppose we apply a field to the muon, with Hamiltonian $\mathcal{H} = \omega S_z$, where S_z is the spin operator in the z direction and ω is a constant. What is the time dependent state and after what time will the state return to its original value? (2 points)

d) Calculate the time dependent expectation value of S_x, $\langle S_x \rangle$. After how much time will $\langle S_x \rangle$ return to its original orientation? Explain the difference from your answer in part c). (3 points)

e) Calculate the time-dependent uncertainty in $\langle S_x \rangle$ given by $\Delta S_x \equiv \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$. At what times is this uncertainty minimized and why? (2 points)

PROBLEM 3: Harmonic oscillator

a) Consider a particle in a one-dimensional harmonic oscillator with Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

where x is the coordinate operator and p is the momentum operator.

a) Sketch the ground state and first excited state wave functions. (1 point)

b) Consider the raising and lowering operators $a^\dagger$, a. Calculate the commutator relationship $[a, a^\dagger]$. (2 points)

c) Let ϕ_n be an eigenfunction of $\hat{N}$ with eigenvalue n, with $\hat{N} = a^\dagger a$. First show that $a\phi_n \propto \phi_{n-1}$ without worrying about the normalization, and then that $a\phi_n = \sqrt{n}\phi_{n-1}$ with proper normalization. Hint: show that $a\phi_n$ is an eigenstate of $\hat{N}$. (2 points)

d) Calculate $\Delta x \Delta p$ for a one-dimensional harmonic oscillator in some generic eigenstate ϕ_n. (2 points)

e) Write down the most general solution to the time dependent Schrodinger equation $\Psi(x,t)$, in terms of harmonic oscillator eigenstates $\phi_n(x)$. (1 point)

f) Using your result from part e) show that the expectation value of x as a function of time can be written as $A\cos\omega t + B\sin\omega t$ where A and B are constants. (2 points)

Note

$$\sqrt{\frac{m\omega}{\hbar}}x\phi_n = \sqrt{\frac{n+1}{2}}\phi_{n+1} + \sqrt{\frac{n}{2}}\phi_{n-1}$$

PROBLEM 4: Time evolution

This problem considers the Schrödinger and Heisenberg pictures.

a) Fill in the following table using the terms "time independent" and "time dependent":

	Schrödinger picture	Heisenberg picture
State ket		
Observable/operator		
Basis ket		

(1.5 points)

b) Let us assume that the Hamiltonian $\mathcal{H}$ of a mass m particle in the Schrödinger picture is given by

$$\mathcal{H} = \frac{\vec{p}^2}{2m} + V(\vec{x}).$$

where $\vec{x}$ and $\vec{p}$ are position and momentum operators, respectively, and $V(\vec{x})$ is some arbitrary potential. Let the system at time $t = t_0$ be in the state $|\alpha, t_0\rangle$. Using the Schrödinger picture, provide an expression for the energy as a function of time t, $t \geq t_0$. Carefully interpret your expression. Using the Heisenberg picture, provide an expression for the energy as a function of time t, $t \geq t_0$. Again, interpret your expression carefully. (3 points)

c) Let us assume now that the Hamiltonian $\mathcal{H}$ of a particle with mass m in the Schrödinger picture is given by

$$\mathcal{H} = \frac{p^2}{2m} + V(x) + \epsilon \sin(\omega_D t),$$

where ϵ and ω_D are constants. Let the system at time $t = t_0$ be in the state $|\alpha, t_0\rangle$. Using the Schrödinger picture, provide an expression for the energy as a function of time t, $t \geq t_0$; carefully interpret your expression. Using the Heisenberg picture, provide an expression for the energy as a function of time t, $t \geq t_0$; carefully interpret your expression. (2 points)

d) How would you expect the energy expectation value for the situation in part b) to change with time? Sketch the time dependence and explain the reasoning behind your plot. (1.5 points)

e) How would you expect the energy expectation value for the situation in part c) to change with time? Sketch the time dependence and explain the reasoning behind your plot. (2 points)

PROBLEM 5: Addition of spin

Consider a system with two spin-1/2 particles.

a) List the complete two particle Hilbert space basis vectors in the direct product basis along with their eigenvalues under the $\vec{S}_1^2$, $\vec{S}_2^2$, S_{1z} and S_{2z} operators. (2 points)

b) The total angular momentum operator is $\vec{S} = \vec{S}_1 + \vec{S}_2$. What is the result of $\vec{S}^2$ acting on each direct product basis vector? (3 points)

c) Construct a new set of basis vectors which are diagonal under the operators $\vec{S}^2$ and S_z. List their eigenvalues under $\vec{S}^2$ and S_z. (3 points)

d) The hyperfine splitting of the ground state of Hydrogen is due to a spin-spin interaction between proton and electron: $H_{HF} = A\vec{S}_p \cdot \vec{S}_e$, where A is some constant. Compute the ground state energy level splitting ΔE_{HF} due to the hyperfine interaction. (2 points)

You may find the following useful:

$$S_\pm = S_x \pm iS_y, \qquad S_\pm|sm\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}|s, (m \pm 1)\rangle$$