Quantum Mechanics
Qualifying Exam - Fall 2024

Notes and Instructions

- *There are five problems but only four problems will count to your grade.* If you chose to solve all five, the problem you score the least will be discarded. Attempt at least four problems as partial credit will be given.
- Write your alias on the top of every page of your solutions. *Do not write your name.*
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3).
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\,, \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathbf{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_2^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: Hydrogen atom

A hydrogen atom has a ground state wavefunction $\psi_{n,\ell,m}$ given by

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}\, \mathrm{e}^{-r/a}.$$

One of the first excited wavefunctions is

$$\psi_{210}(r,\theta,\phi) = \sqrt{\frac{1}{32\pi a^3}}\frac{r}{a}\, \mathrm{e}^{-r/2a}\cos\theta,$$

where $a = 4\pi\epsilon_0\hbar^2/(me^2) = 0.529\times10^{-10}$m is the Bohr radius. Neglect time dependence in the following questions.

a) Find $\langle z\rangle$ for the ψ_{100} and the ψ_{210} states. Express your answer in terms of the Bohr radius a. (2 points)

b) Now find the expectation value $\langle r^2\rangle$ separately in the ψ_{100} and ψ_{210} states. Express your answer in terms of the Bohr radius a. (2 points)

c) Explain *qualitatively* how the apparent size of the electron wavefunction depends on n. (1 point)

d) Find $\langle z\rangle$ for the superposition state

$$\psi_s = \frac{1}{\sqrt{2}}\left(\psi_{100} + \psi_{210}\right).$$

(3 points)

e) Explain the difference between your answers in parts a) and d). (1 point)

f) Using your result in d), estimate the amplitude of the resulting electric dipole moment for the state ψ_s in terms of the electron charge e and the Bohr radius. (1 point)

PROBLEM 2: Spin 1 particle

Consider a spin-1 particle in the state:

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|+\rangle + |0\rangle + |-\rangle \right),$$

where $|+\rangle \equiv |s=1, m_s=1\rangle$, $|0\rangle \equiv |s=1, m_s=0\rangle$, and $|-\rangle \equiv |s=1, m_s=-1\rangle$ are the eigenstates of the operator S_z.

a) Derive the matrix representation of the operators S_z and S_z^2. (1 point)

b) Beginning with the state $|\psi\rangle$, what are the possible measurement outcomes of S_z^2 and the corresponding probabilities? (1 point)

c) Derive the matrix representation of the S_x and S_y operators in the eigenbasis of S_z. (3 points).

d) Assume the Hamiltonian of the system is given by

$$\mathcal{H} = B \left(S_x + S_y \right),$$

where B is a real constant. Find the energies and eigenvalues of $\mathcal{H}$. (3 points)

e) Compute the expectation value of the Hamiltonian in the state $|\psi\rangle$ above. (1 point)

You may find useful that:

$$S_\pm |s, m\rangle = \hbar \sqrt{(s \mp m)(s \pm m + 1)} |s, m \pm 1\rangle$$
$$S_\pm = S_x \pm i S_y$$

PROBLEM 3: Schrodinger and Heisenberg picture

Consider a quantum system described by the Hamiltonian $\mathcal{H}$ and an observable $\mathcal{O}$.

a) Assume observable $\mathcal{O}$ commutes with the time-independent Hamiltonian. Does the expectation value of $\mathcal{O}$ change over time? What about its uncertainty? (2 points)

b) Now consider a spin-$\frac{1}{2}$ particle (an electron) in a time-varying magnetic field along the z-axis,

$$\mathcal{H} = -\frac{h}{2}\gamma B(t)\sigma_z,$$

where σ_z is the Pauli matrix, and

$$B(t) = B_0 \mathrm{e}^{-\alpha t},$$

with γ and α positive constants. If the initial state of the particle at $t = 0$ is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

in the z-basis, find the time evolved ket $|\psi(t)\rangle$. (2 points)

c) Using the Heisenberg picture and the same magnetic field as in part b), calculate the evolution of the operator $\sigma_x(t)$. Hint: use the fact that $\mathrm{e}^{i\theta\sigma_z} = \cos\theta\mathbf{1} + i\sin\theta\sigma_z$. (3 points)

d) Calculate the expectation value $\langle\sigma_x(t)\rangle$ in the $|\psi\rangle$ state using both the Heisenberg and Schrodinger pictures. Are the results consistent? (3 points)

You may find the identities and definitions below useful:

$$\sigma_x\sigma_z = -\sigma_z\sigma_x = -i\sigma_y, \qquad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{1}.$$

Time evolution operator:

$$U(t, t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\mathcal{H}(t')\right].$$

PROBLEM 4: Variational method

Let us consider the ground state energy of a one-dimensional delta function potential

$$V = -\alpha\delta(x),$$

such that the Hamiltonian is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x)\,.$$

To make an estimate for the ground state energy, we can choose a Gaussian trial wave function

$$\psi(x) = Ae^{-bx^2},$$

where b is a constant.

a) Determine the normalization constant A for the wave function $\psi(x)$. (2 points)

b) Evaluate the expectation value of the kinetic energy $\langle T\rangle$. (2 points)

c) Evaluate the expectation value of the potential energy $\langle V\rangle$. (2 points)

d) Determine the best bound for the ground state energy as $\langle\mathcal{H}\rangle_{\min}$ with the parameter b. (3 points)

e) Compare the best bound $\langle\mathcal{H}\rangle_{\min}$ with the exact ground state energy $E_0 = -m\alpha^2/(2\hbar^2)$. (1 point)

PROBLEM 5: Identical particles

Two identical non-relativistic particles of mass m are confined to one dimension (the x axis). Each particle moves in a harmonic trapping potential $V(x) = \frac{1}{2}kx^2$ where $k > 0$.

a) State the allowed single particle energies for a quantum harmonic oscillator. If the two particles are spinless non-interacting bosons, find the ground-state energy E_0 and the first excited state energy E_1 of the system. What is the degeneracy of those states? Justify. (2 points)

b) Solve part a) for spinless non-interacting fermions. (2 points)

c) Assume now that the two-particles are non-interacting spin-$\frac{1}{2}$ electrons. What are E_0 and E_1 and their degeneracies? Justify. (3 points)

For the remainder of this problem, consider an additional attractive interaction which is added to the system Hamiltonian

$$V_{int}(x_1, x_2) = -\alpha k x_1 x_2,$$

where x_1 and x_2 are the particle coordinates and α is a dimensionless parameter $0 < \alpha < 1$.

d) For spinless bosons, find the correction to E_0 to lowest non-vanishing order in α. (3 points)