Quantum Mechanics
Qualifying Exam - Fall 2025

Notes and Instructions

- *There are five problems but only four problems will count to your grade.* If you choose to solve all five, the problem you score the least will be discarded. Attempt at least four problems as partial credit will be given.
- Write your alias on the top of every page of your solutions. *Do not write your name.*
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3).
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathbf{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) \\
Y_2^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: Schrodinger Equation

Suppose a particle of mass m is constrained to exist on a circle (e.g. an electron in a thin wire in the shape of a circular ring) with radius a.

a) Find the energy spectrum and the normalized, time-dependent, eigenstates $\psi_n(\phi, t)$, where ϕ is the polar angle in polar coordinates and n some quantum number. What is the degeneracy of the energy levels? Hint: Use the boundary condition that $\psi_n(\phi, t)$ must be continuous every 2π, and let the potential $V(\phi) = 0$ in the time-independent Schrodinger equation. (3 points)

b) Suppose that at $t = 0$ the particle is known to be in a range from $\phi = -\pi/2$ to $\phi = +\pi/2$, or equivalently

$$\Psi(\phi, t = 0) = \begin{cases} 1/\sqrt{\pi a} & \text{for } -\pi/2 \leq \phi \leq \pi/2 \\ 0 & \text{for } \pi/2 < \phi < 3\pi/2 \end{cases}.$$

Find the representation of this spatial wavefunction in terms of the eigenfunctions, i.e.,

$$\Psi(\phi, t = 0) = \sum_{n=-\infty}^{\infty} c_n \psi_n(\phi, t = 0).$$

(4 points)

c) Now suppose the particle is configured to be in an equal superposition of the ground state and the first two excited states, i.e., $c_0 = c_{\pm 1} = \frac{1}{3}$, and $c_{|n| \geq 2} = 0$. Write down the time-dependent wavefunction and calculate the time-dependent probability density to be found at $\phi = 2\pi/3$. At what times t does this vanish? (3 points)

PROBLEM 2: Time Evolution

An electron with charge e and mass m_e is placed in a uniform magnetic field of strength B pointing in the positive z-direction. The electron spin operator is $\mathbf{S} = (S_x, S_y, S_z)$. The Hamiltonian of the system is

$$\mathcal{H} = \omega S_z,$$

where $\omega = |e|B/m_e c$ is the Larmor frequency.

At time $t = 0$, the electron is in an eigenstate of $\mathbf{S} \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector in the x-z plane, which makes an angle ϕ with respect to the positive z-axis.

a) What is the unitary time-evolution operator $\hat{U}(t)$ for this system in terms of S_z and ω? (1 point)

b) Express the initial, normalized state of the system $|\phi, t = 0\rangle$ in terms of the spin up/down eigenkets $|+\rangle$, $|-\rangle$ of S_z. Do this by acting on the state $|+\rangle$ with the operator for rotation about the y axis: $|\phi, t = 0\rangle = \hat{R}_y(\phi)|+\rangle = e^{-iS_y\phi/\hbar}|+\rangle$. In the spin 1/2 representation, $S_y = (\hbar/2)\sigma_y$, where σ_y is a Pauli matrix. (3 points)

c) What is the $|\phi, t = 0\rangle$ state that minimizes the uncertainty of the measurement of S_x at $t = 0$? Interpret your answer. (2 points)

d) Determine the state $|\phi, t\rangle$ at later times t. What is the probability that a measurement of S_x at time t will yield the eigenvalue $+\hbar/2$? (2 points)

e) What is the expectation value of S_x in this system as a function of time? (2 points)

PROBLEM 3: Spin $1/2$ particle moving in 1D

A particle with mass m moves along one continuous degree of freedom with spatial coordinate x and associated momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$. In momentum representation, this operator satisfies the eigenvalue equation $\hat{p}|p\rangle = p|p\rangle$. The particle has two discrete spin states described by the Hamiltonian operator $\mathcal{H}$:

$$\mathcal{H} = \frac{\hat{p}^2}{2m}\hat{I} + \frac{\hbar k_{so}}{m}\sigma_z\hat{p} + \frac{\Omega}{2}\sigma_x. \tag{1}$$

Here, $\hat{I}$ is the identity operator in spin space, k_{so} and Ω are constants, and σ_x and σ_z are the usual Pauli spin operators for a spin-$\frac{1}{2}$ particle. Hamiltonian $\mathcal{H}$ above couples the spin to spatial degrees of freedom in one dimension.

a) What are the units of k_{so} and Ω? Explain. (1 point)

b) Does operator $\hat{p}$ commute with $\mathcal{H}$? Justify your answer and explain what it implies for the eigenstates and eigenvalues of $\mathcal{H}$. (2 points).

c) Choose a convenient basis to express the Hamiltonian operator $\mathcal{H}$ spanning both the spin space and the momentum one (you should obtain a 2×2 matrix). Explain your reasoning. (1 point)

d) Use your results from parts b) and c) to determine the eigenenergies of $\mathcal{H}$. (3 points).

e) What happens to the energies in the large p limit, where p is the eigenvalue of $\hat{p}$? (1 point)

f) Plot the eigenenergies obtained in d) as a function of p for *i)* vanishing Ω, *ii)* large Ω, and *iii)* small Ω. Explain what the terms 'large' and 'small' mean in this context, i.e., identify the quantity that Ω needs to be compared with in both cases. (2 points)

PROBLEM 4: Time-Dependent Perturbation Theory

Consider a particle confined in a one-dimensional simple harmonic oscillator potential that is perturbed in time. The perturbation begins at $t = 0$ and the particle is initially in the ground state.

a) If the position of the potential well is perturbed (shaken) sinusoidally in time according to $V(x,t) = \alpha\hat{x}\cos(\omega t)$, show that the Hamiltonian can be represented by

$$\mathcal{H} = \hbar\omega_0 \left[\hat{a}^\dagger\hat{a} + \frac{1}{2} + \kappa\left(\hat{a}^\dagger + \hat{a}\right)\cos(\omega t)\right].$$

(1 point)

b) Use first order perturbation theory to approximate the probability of measuring the particle in state $|n\rangle$ as a function of time for $n \neq 0$. Use $2\cos(\omega t) = \mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}$ and neglect the non-resonant terms in the integral. (4 points)

c) Use your results to explain qualitatively the state of the particle at long times, $t \gg 1/\kappa\omega_0$. Hint: Consider the impact of higher order terms in the perturbation theory. (1 point)

d) Consider now the case that instead of modulating the position of the well, we modulate the curvature according to $V(x,t) = \alpha^2\hat{x}^2\cos(\omega t)$. Use first order perturbation theory to determine which transitions are allowed to first order and their resonance frequencies. (4 points)

PROBLEM 5: Spin-Orbit Coupling

A spin-1/2 particle of mass m moves in a central potential $V(r)$. The spin-orbit interaction between the magnetic moment of the particle and the magnetic field that it sees as it moves spatially leads to the Hamiltonian term

$$\mathcal{H}_{\mathrm{LS}} = \frac{1}{2mc^2}\frac{1}{r}\frac{dV(r)}{dr}\,\mathbf{L}\cdot\mathbf{S}, \tag{2}$$

where $\mathbf{L}$ and $\mathbf{S}$ are, respectively, the orbital angular momentum and spin operators of the particle.

In the absence of this term, the energy eigenfunctions of the system may be written in the form

$$R_{nl}(r)Y_l^{m_l}(\theta,\phi)\chi_\pm, \tag{3}$$

where $\chi_\pm$ are two-component spinors describing a spin up/spin down particle, and n is the principal quantum number. These energy eigenfunctions are also eigenfunctions of $\mathbf{L}^2$, L_z, $\mathbf{S}^2$, and S_z.

a) In the absence of $\mathcal{H}_{\mathrm{LS}}$, and for a general potential $V(r)$, the energy depends on n and l. (For the special case of a Coulomb potential, the energy depends only on n.) In the general case, what is the degeneracy of each level? Why is this degeneracy present? What are the possible values of the total angular momentum j of this system? (2 points)

b) The energy shift due to $\mathcal{H}_{\mathrm{LS}}$ may be expressed to first order in (degenerate) perturbation theory as the expectation value of $\mathcal{H}_{\mathrm{LS}}$ with respect to normalized linear combinations of the above degenerate eigenfunctions for which $\mathbf{L}^2$, $\mathbf{S}^2$, $\mathbf{J}^2$ and J_z are diagonal. (You do not need to write these linear combinations explicitly.) Write an expression for the expectation value of $\mathbf{L}\cdot\mathbf{S}$ in each of the possible total angular momentum state. (3 points)

c) Using the result in (b), derive an expression for the energy shift of the energy eigenstates in terms of their orbital angular momentum l and the radial integral

$$\left\langle \frac{1}{r}\frac{dV(r)}{dr}\right\rangle_{nl} \equiv \int_0^\infty R_{nl}^2(r)\frac{1}{r}\frac{dV(r)}{dr}r^2\,dr. \tag{4}$$

How many new energy levels appear? What is the degeneracy of each level? Why does this degeneracy remain? (2 points)

d) Apply this result to the sodium atom for which, in the ground state, the inner 10 electrons fill up all the $n = 1$ and $n = 2$ levels, and the 11th electron is in the $3s$ ($l = 0$) state. Focus on just this electron, moving in an effective potential $V(r)$ created by the nucleus and the inner 10 electrons. Are the $3s$ and $3p$ ($l = 1$) levels degenerate in the absence of $\mathcal{H}_{\mathrm{LS}}$? In the presence of $\mathcal{H}_{\mathrm{LS}}$, into how many levels is the $3p$ level split? What are the possible j values? Assuming that the above radial integral is positive, how are the corresponding energy levels ordered with respect to j? (3 points)