Quantum Mechanics
Qualifying Exam - January 2016

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Harmonic oscillator wave functions

$u_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$

$u_1(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega x^2}{2\hbar}}$

Problem 1: Clebsh-Gordon coefficients (10 pts)

A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

with α a constant and $\mathbf{S}_i$ $(i = 1, 2)$ is the spin operator of the i-th particle.

a) What are the allowed values for the quantum numbers of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? (2 Points)

b) Calculate the energy levels of the Hamiltonian. (2 Points)

c) Let us define the basis of eigenstates of the $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z}, S_{2z} operators, $|s_1 s_2; m_1 m_2\rangle$, where m_1 and m_2 are the quantum numbers of the projection operators S_{1z} and S_{2z} respectively. The system at time $t = 0$ is initially in the state

$$\left| s_1 s_2; \frac{1}{2}, \frac{1}{2} \right\rangle .$$

Find the state of the system at times $t > 0$. (4 Points)

d) Assuming the initial state above, what is the probability of finding the system in the state

$$\left| s_1 s_2; \frac{3}{2}, -\frac{1}{2} \right\rangle$$

at $t > 0$? (2 Points)

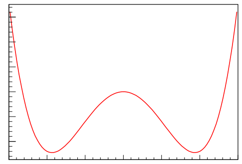

Figure 1: U(x)

Problem 2: Perturbation to a Harmonic Oscillator (10 pts)

Consider a particle of mass, m, moving in a 1-dimensional potential (see Figure 1)

$$U(x) = \lambda x^4 - kx^2.$$

λ and k are positive, and $\lambda \ll \frac{(k^{3/2}m^{1/2})}{4\hbar}$. Approximate the potential near the minima by a simple harmonic oscillator. Here are some useful integrals:

$$\int_{-\infty}^{\infty} x^4 e^{-A(x-a)^2} dx = \frac{1}{4A^{5/2}}(3 + 4a^2 A(3 + a^2 A))\sqrt{\pi}, \text{ for A} > 0$$

$$\int_{-\infty}^{\infty} x^4 e^{-A(x-a)^2} e^{-A(x+a)^2} dx = \frac{3}{16A^{5/2}} e^{-2a^2 A} \sqrt{\frac{\pi}{2}}, \text{ for A} > 0$$

a. Sketch the wavefunctions of the state $|\psi_R\rangle$ which is defined as the state when the particle is found at $x > 0$ and the state $|\psi_L\rangle$ which is the state when the particle is found at $x < 0$. Only consider the lowest energy states near the minima. **(2 Points)**

b. Since the potential is invariant under reflection about the origin, the stationary states must be eigenstates of the parity operator. Express the ground-state and first excited state wavefunctions in terms of $|\psi_R\rangle$ and $|\psi_L\rangle$. **(2 Points)**

c. Estimate the energies of the 2 lowest states using the approximations already described. Hint: use the space representation of the harmonic oscillator wavefunctions and carry out the integrals to find the perturbed energies. **(6 Points)**

Problem 3: Identical particles (10 pts)

Two non-interacting particles of mass m are trapped in a 1-dimensional infinite box of length L situated between $x = 0$ and $x = L$. (In the cases you are considering fermions, assume them to all be spin up.)

(a) [1 points] Write down the single particle energy eigenvalues and wavefunctions.

(b) [1 points] Write down the energy eigenvalues and wavefunctions for two distinguishable particles. Label the states by n_1 for particle 1 and n_2 for particle 2.

(c) [2 points] An energy measurement of the *two identical particle* system yields $E = \hbar^2\pi^2/mL^2$. Write down the state vector/wave function of the system.

(d) [2 points] Suppose instead the energy of the two identical particle system is measured to be $E = 5\hbar^2\pi^2/mL^2$. What is the wave function? *Hint: there are two possibilities.*

(e) [2 points] Show that the fermion state you found in part (d) is an eigenfunction of the Hamiltonian, with the appropriate eigenvalue.

(f) [1 points] Write down the wavefunction for two identical spin-up fermions in the $n_1 = 2$ and $n_2 = 2$ state.

(g) [1 points] If instead you had three particles in the orthonormal states Ψ_1,Ψ_2, and Ψ_3, construct the three particle state for identical fermions.

Problem 4: Matrix Mechanics (10 pts)

Consider a system governed by a Hamiltonian H, with an observable C. The Hamiltonian is represented in the $|e_i\rangle$ basis as:

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Where } |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of H are

$$|E_1 = -\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, |E_2 = \hbar\omega, 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, |E_2 = \hbar\omega, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let C be represented in the $|e_i\rangle$ basis as

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

At t=0, the system is in the state: $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

a) At time t=0, the observable C is measured. What results are possible and with what probabilities? (2 pts)

b) Determine the representation of the time evolution operator $U(t, t_0 = 0)$ in the $|e_i\rangle$ representation. (2 pts)

c) Determine $|\Psi(t)\rangle$ in the $|e_i\rangle$ basis. (2 pts)

d) If C is measured at some later time t, what results are possible and with what probabilities? (2 pts)

e) Are your probabilities time dependent or time independent? Explain (2 pts)

Problem 5: Magnetic Moments and Spin (10 pts)

Consider a spin 1/2 particle with a magnetic moment. We can write the interaction between the spin and an external magnetic field using the Hamiltonian:

$$H = -\gamma \vec{B} \cdot \vec{S} \tag{1}$$

where $\vec{B}$ is the external field, $\vec{S}$ is the spin operator for the particle, and γ is a real positive constant. In this problem, use the usual basis states that are eigenstates of S_z

$$S_z \chi_\pm = \pm \frac{\hbar}{2} \chi_\pm, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

For this problem, assume the magnetic field lies in the x-z plane:

$$\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z \tag{3}$$

(a) [1 pt] Solve for the eigenenergies for the Hamiltonian, showing your work. Explain the physics of your results.

(b) [2 pts] Any state of the spin can be written in the $\chi_\pm$ basis as:

$$\Psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \tag{4}$$

Using the Hamiltonian, derive the first-order coupled differential equations that give the time dependence for $\alpha(t)$ and $\beta(t)$. In other words, derive the equations for $\dot{\alpha}(t)$ and $\dot{\beta}(t)$.

(c) [2 pts] Show that you can re-write your results from part (b) as two uncoupled second-order differential equations:

$$\begin{aligned} \ddot{\alpha}(t) &= -\frac{\gamma^2 B_T^2}{4} \alpha(t) \\ \ddot{\beta}(t) &= -\frac{\gamma^2 B_T^2}{4} \beta(t) \end{aligned} \tag{5}$$

where $B_T = \sqrt{B_x^2 + B_z^2}$ is the magnitude of the total magnetic field. How is this result related to what you found in part (a)?

Of course, the solutions to these equations are:

$$\begin{aligned} \alpha(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ \beta(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{aligned} \tag{6}$$

with $\omega = \frac{\gamma B_T}{2}$.

(d) [3 pts] Consider the situation where the spin is in the spin-up S_z state χ_+ at time $t = 0$. Using the boundary conditions at time $t = 0$, determine the values for the constants $C_1,\ C_2,\ C_3,\ C_4$ that will solve for the time-dependence of the state. Remember that the equations in part (c) are second-order, so you need two boundary conditions at $t = 0$ for each.

(e) [2 pt] Write down the time-dependent probabilities, $P_\pm$ of the spin being in the spin-up and spin-down S_z states. Show that your results are correct in the two cases where $B_x = 0$ and $B_z = 0$.

Problem 6: Electron in a Finite Square Well (10 pts)

Consider an electron of energy E incident from x=$-\infty$ on a symmetric one-dimensional square well of depth V_0 and width L.

$$V(x) = \begin{cases} 0, & \text{x < -L/2} \\ -V_0, & \text{-L/2 < x < L/2} \\ 0, & \text{x > L/2} \end{cases}$$

a) Write down the solutions to the time-independent Schrodinger Equation for this situation. There should be five integration constants (2 points)

b) Apply boundary conditions to find the probability that the electron is transmitted past the finite well (4 points)

c) For what values of E is there a 100% probability for transmission past the well? (2 points)

d) Consider a potential well with V_0 large enough for there to be two bound states. For this well, what is the smallest electron energy (E > 0) for which there is a 100% probability for transmission? Your answer will depend on V_0 and other parameters in the problem. (2 points)