Quantum Mechanics
Qualifying Exam – January 2022

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work**.

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^\dagger), \quad \left[\hat{a},\hat{a}^\dagger\right] = 1, \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle, \quad \text{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\ . \end{aligned}$$

The Hermite polynomials:

$$\begin{aligned} H_0(y) &= 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2-2 \\ H_n(y) &= (-1)^n e^{y^2}\frac{\partial^n}{\partial y^n}e^{-y^2} \end{aligned}$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \qquad Y_2^{\pm 2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$Y_1^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \quad Y_2^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

Angular momentum raising and lowering operators:

$$\begin{aligned} L_\pm &= L_x \pm i\,L_y \\ L_+|\ell,m\rangle &= \hbar[\ell(\ell+1)-m(m+1)]^{1/2}|\ell,m+1\rangle \\ L_-|\ell,m\rangle &= \hbar[\ell(\ell+1)-m(m-1)]^{1/2}|\ell,m-1\rangle \end{aligned}$$

Gaussian Integral:

$$I_0(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = (\pi/\alpha)^{1/2}\ , \qquad \alpha > 0$$

where α is usually chosen to be real.

PROBLEM 1: Harmonic Oscillator

Consider a one-dimensional quantum harmonic oscillator of mass m and angular frequency ω. Its Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \,,$$

where $\hat{x}$ is the coordinate and $\hat{p}$ is the momentum operator.

The annihilation operator is given by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}$$

(a) (1 point) Show that $[\hat{a}, \hat{a}^\dagger] = 1$.

(b) (2 points) Calculate $\langle \hat{H} \rangle$, $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, $\langle \hat{x}^2 \rangle$, and $\langle \hat{p}^2 \rangle$ in the eigenstate $|n\rangle$.

(c) (2 points) Consider the state vector at time $t = 0$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Find the state vector $|\psi(t)\rangle$ at time t (expressed in terms of $|0\rangle$ and $|1\rangle$) and $\langle \hat{x} \rangle$ and $\langle \hat{x}^2 \rangle$.

(d) (2 points) Consider the state vector at time $t = 0$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$$

and find $\langle \hat{x} \rangle$ as a function of t.

(e) (3 points) A un-normalized energy eigenfunction of the harmonic oscillator can be written as

$$\psi_a(x) = (2x^3 - 3x)e^{-x^2/2} \,.$$

Calculate two other un-normalized eigenfunctions which are closest in energy to ψ_a. Show all work. (To make the problem easier to solve, you can let $\hbar = m = \omega = 1$)

PROBLEM 2: Spin-$\frac{1}{2}$ Interferometer

(a) (2 points) A single spin 1/2 particle has been measured to point along the x-axis with eigenvalue $s_x = 1/2$. What is the probability that a subsequent measurement would find the spin to be parallel to the z-axis?

(b) (2 points) Suppose that instead of measuring, you introduce a device that can spatially separate $|s_x = 1/2\rangle$ according to s_z components, sending $|s_z = +1/2\rangle$ upward and $|s_z = -1/2\rangle$ downward. Assuming the two s_z components are now spatially separated far enough that the wavefunctions do not overlap, describe what a s_x measurement would find in this situation, both in terms of spin and spatial location.

(c) (3 points) Now imagine that instead of measuring the separated components, we introduce a mechanism that causes a rotation about $\hat{z}$ on **only** $|s_z = +1/2\rangle$, without measuring it, according to

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} .$$

What are the probability *amplitudes* of $|s_z = +1/2\rangle$ and $|s_z = -1/2\rangle$ following this operation?

(d) (3 points) After the rotation with $R_z(\theta)$, let us bring the two components back to the same spatial location, perfectly overlapping the state vectors and the wavefunctions. Calculate the probability of measuring $s_x = 1/2$ as a function of θ.

PROBLEM 3: Identical particles

Consider two non-relativistic point particles, each with mass m, in one-dimensional space interacting through the δ-function potential

$$V(z_1 - z_2) = g\delta(z_1 - z_2), \tag{1}$$

where g is a constant and z_1 and z_2 denote the position coordinates of particle 1 and particle 2, respectively.

Note: Parts (a)-(d) do NOT consider the spin degree of freedom.

(a) (0.5 points) What units does the constant g have?

(b) (2.5 points) Write down the two-particle Hamiltonian $\hat{H}$. Using the relative coordinate z,

$$z = z_1 - z_2, \tag{2}$$

and the center-of-mass coordinate Z,

$$Z = \frac{z_1 + z_2}{2}, \tag{3}$$

rewrite the two-particle Hamiltonian $\hat{H}$ in terms of the relative and center-of-mass coordinates, i.e., write $\hat{H}$ as $\hat{H}_{rel} + \hat{H}_{cm}$, where $\hat{H}_{rel}$ and $\hat{H}_{cm}$ denote, respectively, the center-of-mass and relative Hamiltonian, and argue concisely or show that the relative and center-of-mass degrees of freedom separate.

(c) (1.5 points) Determine the complete set of eigenstates for the center-of-mass Hamiltonian.

(d) (3 points) This part of the problem considers the relative Hamiltonian. We want to determine whether or not $\hat{H}_{rel}$ supports odd-parity bound states. To address this problem, define the terms "even parity" and "odd parity." Then define concisely what the condition for a bound state is. Last, argue why $\hat{H}_{rel}$ does not support an odd-parity bound state.

(e) (2.5 points) Let us denote the even-parity bound state of $\hat{H}_{rel}$ by $\psi_{even}^{bound}(z_1 - z_2)$; there exists exactly one such bound state for negative g (you do NOT have to derive this!). Assuming that the two particles are spin-1/2 fermions, write down the lowest energy eigen state for negative g for a spin-singlet and for a spin-triplet state; in doing so, account for the center-of-mass, relative, and spin degrees of freedom.

PROBLEM 4: Stationary Perturbation Theory

The energy eigenstates of a particle of mass m in a one dimensional potential satisfy

$$\hat{H}\Psi_n(x) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0(x) + V_1(x)\right\}\Psi_n(x) = E_n\Psi_n(x)$$

where

$$V_0(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

where L is a positive constant, and

$$V_1(x) = U_1\ \delta\left(x - \frac{L}{2}\right)$$

where $\delta(x)$ is a Dirac delta function and $U_1 \geq 0$ is a constant giving the strength of the potential. We will look at this problem treating $V_1(x)$ as a perturbation.

(a) (1 point) In the unperturbed ($U_1 = 0$) case, what are the eigenenergies and normalized energy eigenfunctions? Denote these by $E_n^{(0)}$ and $\Psi_n^{(0)}(x)$ respectively.

(b) (2 point) Solve for the first order correction to the ground state energy, $E_1^{(1)}$.

(c) (3 point) Solve for the first order correction to the ground state wavefunction, $\Psi_1^{(1)}(x)$.

(d) (3 point) Solve for the second order correction to the ground state energy, $E_1^{(2)}$.

(e) (1 point) Sketch the exact ground state wavefunction for when the perturbation is moderately strong compared to the ground state energy, $U_0 L \approx E_1^{(0)}$. Your answer need not be numerically precise, but should be qualitatively correct.

Your final expressions in your answers above may involve sums, but should not involve any unevaluated integrals.

PROBLEM 5: Generalized Uncertainty Principle

The normalized wave function of a one-dimensional particle is

$$\langle x|\psi\rangle = \psi(x) = N\left[e^{\lambda x}\theta(-x) + e^{-\lambda x}\theta(x)\right]$$

for some real parameter $\lambda > 0$ and state vector $|\psi\rangle$. Here $\theta(x)$ is the step function and the normalization constant N is real and positive.

(a) (1 point) What is the normalization constant N?

(b) (2 point) Calculate the expectation values of $\hat{x}^2$: $\langle\hat{x}^2\rangle$.

(c) (2 point) Find the momentum space wave function $\phi(p)$

$$\phi(p) \equiv \langle p|\psi\rangle\ .$$

(d) (2 point) Calculate the expectation values of $\hat{p}^2$: $\langle\hat{p}^2\rangle$.

(e) (3 point) Determine the uncertainty relation $\Delta x\Delta p$ with $\langle\hat{x}\rangle = 0$ and $\langle\hat{p}\rangle = 0$.

PROBLEM 6: Angular Momentum Operator

Consider the standard angular momentum operators and basis states:

$$L^2|\ell, m_z\rangle = \ell(\ell+1)\,\hbar^2|\ell, m_z\rangle, \qquad L_z|\ell, m_z\rangle = m_z\hbar|\ell, m_z\rangle \tag{4}$$

The angular momentum raising and lowering operators are very useful for calculations of angular momentum properties. These are defined as:

$$L_\pm = L_x \pm iL_y \tag{5}$$

$$L_\pm|\ell, m_z\rangle = \sqrt{\ell(\ell+1) - m_z(m_z \pm 1)}\,\hbar|\ell, m_z \pm 1\rangle \tag{6}$$

For this problem, consider the sub-space of Hilbert space with $\ell = 1$. The L_z basis states can be defined as $|m_z\rangle$ with $m_z = 1,\ 0,\ -1$

(a) (2 point) Using the raising and lowering operators, show that $\langle m_z|L_x|m_z\rangle = 0$ and $\langle m_z|L_y|m_z\rangle = 0$.

(b) (2 points) Calculate the matrix elements $\langle m_z|L_\pm|n_z\rangle$ of the raising and lower operators for all m_z and n_z states.

Use your answers to write down the 3×3 matrix representations of these two operators.

(c) (1 point) Show from your previous results that the raising and lowering operators are non-Hermitian but the operators L_x and L_y are (of course) Hermitian.

(d) (2 points) Solve for the eigenvalues and eigenvectors of the operator L_y.

(e) (3 points) Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{5}}\left(|m_z = 1\rangle + 2|m_z = 0\rangle\right) \tag{7}$$

What are the possible outcomes and probabilities for a measurement of L_y for the state $|\psi\rangle$?