Quantum Mechanics
Qualifying Exam - Spring 2025

Notes and Instructions

- *There are five problems but only four problems will count to your grade.* If you choose to solve all five, the problem you score the least will be discarded. Attempt at least four problems as partial credit will be given.
- Write your alias on the top of every page of your solutions. *Do not write your name.*
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3).
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathbf{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_2^{\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: Harmonic Oscillator

Consider a one-dimensional quantum harmonic oscillator of mass m and frequency ω. We denote by $|n\rangle$ (where $n = 0, 1, 2, \ldots$) the n-th excited eigenstate of the oscillator. Its Hamiltonian is given by

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2,$$

where $\hat{x}$ ($\hat{p}$) is the position (momentum) operator. The annihilation operator is given by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}.$$

a) Show that $[\hat{a}, \hat{a}^\dagger] = 1$. (1 point)

b) Calculate the first 3×3 block of matrix elements representing $\hat{a}$ and $\hat{a}^\dagger$ in the basis $|n\rangle$ (i.e. the upper-left corner of each matrix). (2 points)

c) Calculate the expectation values $\langle n|\hat{\mathcal{H}}|n\rangle$, $\langle n|\hat{x}|n\rangle$, and $\langle n|\hat{x}^2|n\rangle$ (3 points)

d) Consider the following normalized superposition at time $t = 0$:

$$|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right).$$

Find the evolved ket state $|\psi(t)\rangle$ (in terms of $|0\rangle$ and $|1\rangle$), $\langle \hat{x}(t)\rangle$, and $\langle \hat{x}^2(t)\rangle$ as a function of time t. (3 points)

e) Consider the following normalized superposition at time $t = 0$:

$$|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |2\rangle\right).$$

Find $\langle \hat{x}(t)\rangle$ as a function of time t. (1 point)

PROBLEM 2: Variational Method

The variational method is often used for the determination of the ground-state energies when systems are not exactly solvable. Using a normalized trial wave function, an estimate of the ground state energy can be found.

a) Given that ψ is properly normalized, show that $\langle\psi|\hat{\mathcal{H}}|\psi\rangle \geq E_o$, where E_o is the true ground state energy. $\hat{\mathcal{H}}$ is the Hamiltonian of the system. (2 points)

b) Use the trial wave functions

$$\psi(r, \theta, \phi, \alpha) = \frac{1}{N} e^{-\alpha r^2}$$

where r, θ, ϕ are spherical coordinates, α is the variational parameter, and N is a normalization constant. What is the orbital angular momentum quantum number ℓ for the wave function? Explain why this value of ℓ is a good choice for the trial wave function. (1 point)

c) Compute $E(\alpha) = \langle\psi|\hat{\mathcal{H}}|\psi\rangle$ as a function of α. To simplify the solution, note that $\langle\psi|\hat{T}|\psi\rangle = 3\hbar^2\alpha/(2m)$ for the expectation value of the kinetic energy operator $\hat{T}$. Determine $E(\alpha)$ for the potential energy $V = -e^2/r$. (3 points)

d) Using your result from part c) determine the best estimate of the ground-state energy of the hydrogen atom using the trial wave function shown above. Express your result in terms of e, m and $\hbar$. Verify that your best estimate provides a minimum energy. Compare your result to the exact ground state energy $-me^4/(2\hbar^2)$. (4 points)

Potentially useful integrals:

$\int_0^\infty e^{-br^2} dr = \frac{1}{2}\sqrt{\frac{\pi}{b}}$

$\int_0^\infty re^{-br^2} dr = \frac{1}{2b}$

$\int_0^\infty r^2 e^{-br^2} dr = \frac{1}{4b}\sqrt{\frac{\pi}{b}}$

PROBLEM 3: Identical particles

Where appropriate, make sure that the wave function has the proper symmetry under the exchange of identical particles. This problem treats the nucleus as an infinitely heavy positively charged point particle. The only interaction at play is the Coulomb interaction. Electrons are treated as spin-$\frac{1}{2}$ particles and relativistic effects are neglected.

a) What are the eigenenergies of the hydrogen atom ($Z = 1$), He^+ ion ($Z = 2$), and C^{5+} ($Z = 6$)? Hint: You can use the Bohr model energy for the hydrogen atom. (2 points)

b) Neglecting the electron-electron repulsion, what is the ground state energy and the ground state wave function of the (neutral) helium-atom He? The helium atom has two electrons. In addressing this question, assume that the wavefunction of each electron has the form $\psi_{\{\alpha\}}(\mathbf{r})\chi_\sigma$, where $\{\alpha\}$ labels the set of quantum numbers that specify the orbital part of the electron wavefunction $\psi_{\{\alpha\}}(\mathbf{r})$, and χ_σ is the spin part. (2 points)

c) Neglecting the electron-electron repulsion, construct the excited wave functions and associated energy spectrum of the (neutral) helium atom when one of the two electrons sits in the lowest possible energy state. Explicitly comment on the symmetry of the spin part of the wavefunction. (3 points)

d) Neglecting electron-electron interactions, what is the ground state wave function of the neon atom ($Z = 10$)? Explain. Hint: The ground state wave function can be conveniently written as a Slater determinant. (3 points)

PROBLEM 4: Addition of Angular Momenta

a) Consider a system of 2 particles with angular momentum quantum numbers $j_1 = 2$ and $j_2 = \frac{1}{2}$. Without using a Clebsch-Gordan coefficient table, express the total angular momentum states $|j, m\rangle = |\frac{5}{2}, \frac{3}{2}\rangle$ and $|\frac{3}{2}, \frac{3}{2}\rangle$ in terms of product states. (2 points)

b) If the particles in part a) are in the states $|j_1, m_1\rangle = |2, 1\rangle$ and $|j_2, m_2\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ what is the probability that the total angular momentum is in the state $|\frac{3}{2}, \frac{3}{2}\rangle$? (2 points)

c) Quarks carry spin $\frac{1}{2}$. Three quarks bind together to make a baryon and two quarks bind together to form a meson. Assume the quarks are in the ground state. What spins are possible for baryons? What spins are possible for mesons? Explain your answers. (2 points)

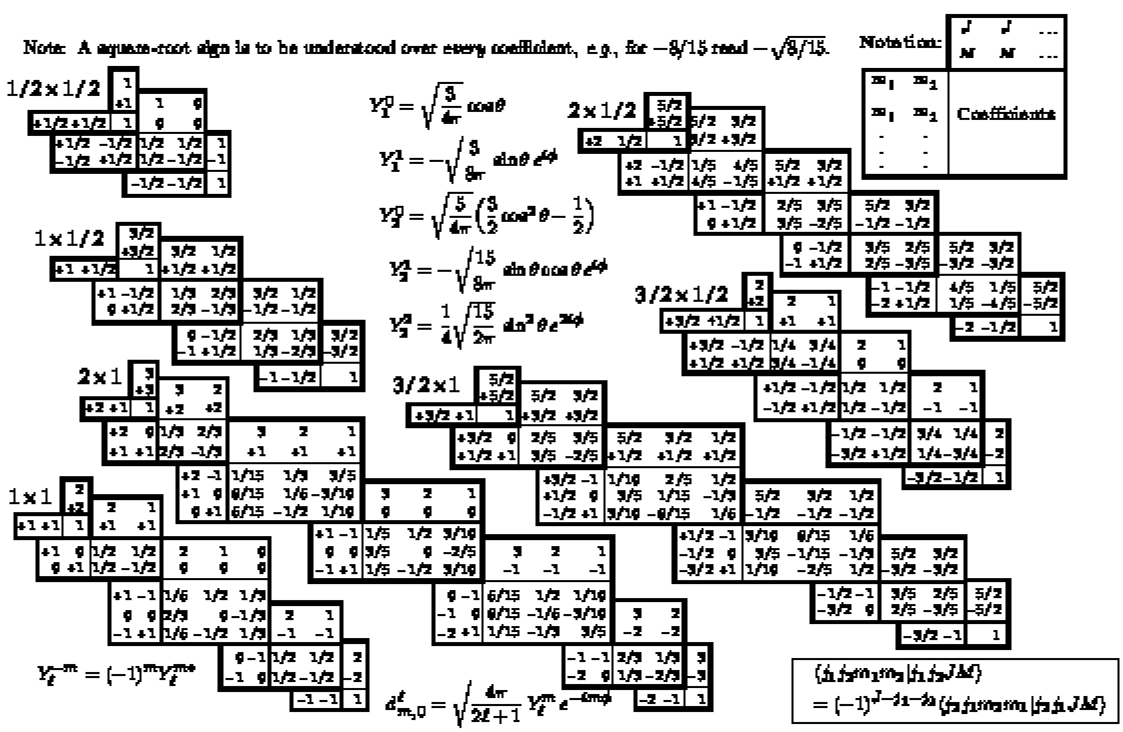
d) A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the approximate Hamiltonian $\hat{\mathcal{H}} = \alpha \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$, with α a given constant and $\hat{\mathbf{S}}_i$ $(i = 1, 2)$ the spin operator of particle i. The system at time $t = 0$ is in the following eigenstate of $\hat{S}_1^2, \hat{S}_2^2, \hat{S}_{1z}, \hat{S}_{2z}$:

$$|s_1, m_1; s_2, m_2\rangle = \left|\frac{3}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle.$$

What is the probability of finding the system in the state $|\frac{3}{2}, \frac{3}{2}; \frac{1}{2}, -\frac{1}{2}\rangle$ for times $t > 0$. (4 points)

Possibly useful information:

$$\hat{J}_-|j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}|j, m-1\rangle$$
$$\hat{J}_+|j, m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}|j, m+1\rangle$$



PROBLEM 5: Coherent States

Consider a one-dimensional harmonic oscillator with mass m and frequency ω that is described by the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively. These operators satisfy the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

a) If α is a complex number, show that the unitary operator $\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$ translates the annihilation operator $\hat{a}$ by α, i.e.

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha. \tag{1}$$

Hint:

$$e^{\hat{A}}\hat{X}e^{-\hat{A}} = \hat{X} + \left[\hat{A}, \hat{X}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{X}\right]\right] + ... \tag{2}$$

for any two operators $\hat{A}$ and $\hat{X}$. (2 points)

b) Consider a "coherent state" defined as $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where $|0\rangle$ is the oscillator's ground state. Write this state in the basis of energy eigenstates $|n\rangle$ $(n = 0, 1, 2, ...)$. Hint: For two operators $\hat{A}$ and $\hat{B}$ whose commutator is a complex number, we have

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}. \tag{3}$$

Apply this formula for $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\alpha^*\hat{a}$. (3 points)

c) Show that the coherent state $|\alpha\rangle$ defined in (b) is an eigenstate of the operator $\hat{a}$. Find its eigenvalue. Hint: You can solve this part by using either (a) or (b). (2 points)

d) Calculate the time dependence of a state initially prepared in the coherent state $|\alpha\rangle$. Show that this state is (up to a phase factor) a coherent state $|\alpha(t)\rangle$ and write an explicit expression for $\alpha(t)$. (3 points)