Abstract

By introducing financing constraints and firm competition into the real option framework, we derive important implications for firm hedging policies, resulting cash flow risk, and the expected stock returns. First, contrary to the casual intuition, we show that safer firms may hedge more. This is because real options in risky firms induce a positive correlation between cash flows and investment demand. Second, consistent with empirical evidence, firms with a higher component of the firm-specific risk hedge less and have more valuable growth options. Third, the model produces a positive correlation between the book-to-market ratio and firm’s market beta, contributing to the explanation of the “value puzzle”. This result does not require that firms have financial or operating leverage.

Keywords: real and financial hedging, investment options, competition, idiosyncratic risk, value premium.

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Corporate risk management, broadly defined, can take two main forms. First, a company can avoid undertaking risky investment altogether, diversify firm’s activities, or outsource and design production process in a way that reduces its overall risk exposure (we refer to it as “real hedging”). Second, a firm can use financial derivatives, such as commodity and interest rates instruments, to reduce the negative consequences of lower-tail profit outcomes (“financial hedging”). The academic literature focuses almost exclusively on this second form of risk management. Implicit in virtually all of this research is the assumption that firm assets’ risk and the type of risk are exogenous. Thus the literature overlooks the fact that corporate risk management is a “strategic activity that encompasses everything from operating changes to financial hedging to the buying and selling of plants and new businesses—anything that affects the level and variability of cash flows going forward.”

In this paper, we begin to fill this gap by turning attention to the effect of a firm’s real investment on its financial hedging policy and the resulting cash flow risk. We recognize that, in addition to costs, asset risk has benefits because it increases the value of firm’s investment options. This idea has long been accepted in the real options literature, but has largely been ignored in the hedging literature. We also recognize that hedging can affect the size of financing costs for investment options when investment demand is positively correlated with cash flows. Building on this insight, we find that firms with safer assets may engage in more financial hedging, and that the hedging policy depends on the composition of firm’s asset risk.

Using a contingent claims framework, we build a model that features optimally exercised investment option, two sources of risk (systematic and firm-specific), dynamic hedging, and product market competition. The model is

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most closely related to Caballero and Pindyck (1996), with the exception that we introduce financing constraints and allow for hedging. Competition within industry implies that the upside of the systematic component of the profit is limited by a reflecting barrier, and therefore firms’ investment strategies are more sensitive to the firm-specific demand shocks. We offer an extension of the model that allows for the incremental investment and continuously paid operating costs.

The model demonstrates that, contrary to the view entertained in the literature, firms that invest in safer assets choose to hedge more. The intuition is based on two observations. First, absent investment options, constrained firms choose high hedging ratios since doing so provides enough cash to meet their operating expenses in all states of the world. Second, real options reduce incentives to hedge because hedging lowers the correlation between firm’s investment demand and available funds. Since the value of growth options increases with volatility, the first effect dominates for the low-risk firms and they choose to hedge more. Our model thus extends and complements the results in Froot, Scharfstein, and Stein (1993) to accommodate the endogenous investment opportunities and the heterogeneity of asset risk.

Our analysis in the second part of the paper draws on the distinction between the systematic and idiosyncratic components of the firm’s risk. The idea is that the value of future investment options is mainly derived from the firm’s unique assets and is thus related to the idiosyncratic component of the firm’s asset risk. For example, if the investment opportunities improve uniformly for all firms in industry, the increase in competition associated with higher production and new entry into the market will remove some of the firm’s profits (see, e.g., Dixit and Pindyck (1991) and Grenadier (2002) for the discussion of the effect of competition on real options). However, when the firm’s success is unique, its
real options are likely to increase in value, implying that firms which start with more unique assets derive a larger component of value from real options.

Using this insight, we derive two additional results. First, we find that firms with a larger proportion of unique risk use lower hedging ratios. This finding sheds light on the empirically observed differences between hedging policies of growth firms and value firms. For example, there is strong empirical evidence that large profitable firms with fewer growth opportunities (low market-to-book ratios) tend to hedge more (see e.g., Mian (1996) and Bartram, Brown, and Fehle (2009)). Our approach provides an explanation to this empirical regularity that is not based on the cost of hedging.

Second, we provide a new risk-based explanation for the value anomaly documented by Fama and French (1992). In the model, the firms that are born with a higher proportion of idiosyncratic risk, and therefore lower betas, have more valuable investment options. This effect is responsible for the positive correlation between equity betas and book-to-market ratios. However, even if firms are initially identical, the cross-sectional correlation arises as the firm-specific shocks evolve. This is because firms exercise their options when the idiosyncratic component of the profitability shock is large (the systematic component is fixed in the cross-section). Such firms have a larger proportion of the idiosyncratic assets and exercised their options, therefore they have smaller betas. At the same time, because of the relatively high value of the profitability shock they also have lower book-to-market ratio.

As suggested by the existent literature (e.g., Gomes, Kogan, and Zhang (2003), Cooper (2006)), the correlation between the book-to-market equity ratio and firm’s equity beta alone is not sufficient to generate the empirically observed “value” effect because the cross-sectional tests typically control for equity betas. Specifically, the conditional CAPM holds in our setting with a
single risk factor, and thus equity beta is a sufficient statistic for the expected stock returns. Nonetheless, the betas are likely to be mismeasured either because the tests fail to use the conditioning information or because the proxy for the market portfolio is imperfect. Our analysis of simulated data indicates that the model can generate a positive relation between book-to-market ratios and expected stock returns when unconditional betas are controlled for. Our results are consistent with Jagannathan and Wang (1996), who find that a conditional CAPM model, in which betas and the market risk premium depend on conditioning information, can empirically account for a large part of the explanatory power of book-to-market in the cross-section of stock returns.

The paper is organized as follows. Section 1 offers the review of the relevant literature. Section 2 presents a model with a single investment option and the operating cost payable at a fixed point of time. Section 3 extends the model to the infinite number of investment options and continuously paid operating costs. Section 4 discusses the simulation results and the empirical predictions. The last section provides concluding remarks.

1 Literature Review

A number of papers establish that financial hedging creates value. For example, Smith and Stulz (1985) argue that hedging can reduce the expected bankruptcy costs and tax; Graham and Smith (1999) document that hedging minimizes tax bill for approximately 75% of firms in the United States; Leland (1998) and Graham and Rogers (2002) show that hedging increases firm’s debt capacity; DeMarzo and Duffie (1995) conclude that hedging can improve stock price informativeness and reduce information asymmetry; Mackay and Moeller (2007) document that hedging adds 2% to 3% in value because revenues and costs are nonlinearly related to prices; and Morellec and Smith (2007) show that hedging
can mitigate managers’ overinvestment incentives. With exception of Morellec and Smith (2007), real investment is held fixed in all of these papers.

Several papers use structural models to derive the optimal hedging ratio. In a closely related paper, Froot, Scharfstein, and Stein (1993) argue that hedging can improve firm’s ability to undertake investment when external finance is costly, but firms should not fully hedge when investment opportunities are positively correlated with cash flows. Our paper extends their results by linking hedging to the risk and type of investment. Adam, Dasgupta, and Titman (2007) recognize that a firm’s risk management choice is affected by the strategies of other firms in the industry and show that an individual firm may not hedge if most of its competitors hedge. In their paper, the firm’s goal is to get financing into the states of the world where firm’s competitors lack financing. The results in our paper that stem from the competition have a different flavor because, following real options literature, we allow free competitor entry conditional on the realization of profit shock. Fehle and Tsyplakov (2005) show that in the presence of costs of financial distress and fixed costs of hedging, firms that are either far away from distress or close to distress may choose not to hedge. Fehle and Tsyplakov do not consider investment or choice of risk. Bolton, Chen, and Wang (2009) use a structural model with investment to determine the incentives to hedge, but do not consider how the risk of real investment and degree of financing constraints affects the hedging strategy.²

Our paper is related to the literature that examines effects of competition on value and exercise strategy of real options. For example, Grenadier (2002) argues that competition erodes value of real options and reduces the advantage to waiting to invest. In contrast, Leahy (1993) and Caballero and Pindyck (1996) argue that despite the fact that option to wait is less valuable in a competitive

²The optimal hedging ratio in their model is determined by the convexity of the value function and transactions costs, but is independent of financial constraints or value of real options.
environment, irreversible investment is still delayed because the upside profits are limited by new entry. By focusing on nonlinear production technology, Novy-Marx (2007) shows that firms in a competitive industry may delay irreversible investment longer than suggested by a neoclassical framework. Aguerrevere (2009) considers how competition within industry affects the timing of real option exercises when firms face market-wide uncertainty. By introducing operating leverage into the model, Aguerrevere derives the implications of competition and variation in industry demand for expected stock returns. None of these papers, however, analyze the investment risk and hedging incentives, which is the main focus of our paper.

Our paper also contributes to the rapidly growing literature that links the theory of investment under uncertainty to the determinants of the cross-section of stock returns. Berk, Green, and Naik (1999) were among the first to link the number of investment options to firm’s risk. Carlson, Fisher, and Gianmarino (2004) model investment problem with operating leverage and show that asset betas vary over time with investment. Their model can account for the book-to-market and size anomalies. Using a framework with costly investment reversibility and countercyclical price of risk, Zhang (2005) shows that in bad times assets in place are riskier than growth options and hence should command risk premium (“value anomaly”). Cooper (2006) develops similar intuition and obtains the value effect in a model that allows for investment lumpiness and the constant price of risk. All of these papers, however, ignore the asset-pricing implications of product market competition and do not analyze financial hedging. The value effect in our paper appears even when the price of risk is constant and firms have no operating leverage.

Gomes, Kogan, and Zhang (2003) build a general equilibrium model with perfect competition and demonstrate the value effect in the cross-section. The
important difference between our model and theirs is that we attribute the book-to-market effect to the cross-sectional differences in the riskiness of growth options, whereas in their model this effect is driven by differences in riskiness of assets-in-place. Another difference is that we consider the optimal timing of investment and study the effects of competition on the value of investment options. Although Gomes, Kogan, and Zhang model a perfectly competitive market, there are no preemptive investment motives in their model because projects arrive continuously, and if not taken, disappear immediately.

2 The Model with a Single Investment Option

This section lays out a simple model with a single investment option. Each firm in the economy produces one unit of output that can be sold at a price $P_i$ (time subscript is suppressed)

$$P_i = (1 - \rho) X_i + \rho Y Q^{-\varepsilon},$$  \hspace{1cm} (1)

where $X_i$ is the firm-specific demand shock (e.g., the tastes for the differentiated firm’s product change), $Y$ is the systematic demand shock that affects the whole industry, $Q$ is the number of firms in the industry, $1/\varepsilon$ is the positive price elasticity of demand, and $\rho$ measures the correlation between the product price and the systematic demand shock.

The profitability shocks follow geometric Brownian motions in the risk-neutral measure

$$dX_i = \mu_x X_i dt + \sigma_x X_i d\zeta_i,$$  \hspace{1cm} (2)

$$dY = \mu_y Y dt + \sigma_y Y d\zeta_y,$$  \hspace{1cm} (3)
where \(dz_i\) and \(dz_y\) are the increments of the uncorrelated standard Wiener processes, \(E[dz_i dz_y] = 0\). The firm-specific shocks have identical drifts and volatilities and are uncorrelated, \(E[dz_i dz_k] = 0\) for \(i \neq k\).

We model the product market competition by assuming that new firms can enter the industry by paying a fixed cost \(R\). Whereas the value of shock \(Y\) is common knowledge prior to the entry, the prospective entrants cannot observe the values of their idiosyncratic shocks until they pay an entry cost and get a random draw of \(X_i\). For simplicity, we adopt the assumption that firms in the market are identical (with the exception of the asset volatility and the history of firm-specific shocks), competitive, risk neutral, and infinitesimally small. This assumption allows us to treat firms as price-takers and to ignore the effect of firm’s own output on the equilibrium price.

Since for tractability purposes we do not model optimal firm exit, we assume that the number of firms in the industry decays over time with intensity \(\lambda\)

\[
dQ_t = -\lambda Q_t dt. \tag{4}
\]

By denoting \(y = \rho Y Q^{-\varepsilon}\) and \(x_i = (1 - \rho) X_i\) and using the Ito’s lemma, we can write the dynamics of the processes \(x_i\) and \(y\) when no new entry takes place as

\[
\begin{align*}
dx_i &= \mu x_i dt + \sigma x_i dz_i, \tag{5} \\
dy &= (\mu_y + \varepsilon \lambda) y dt + \sigma_y y dz_y, \tag{6}
\end{align*}
\]

where the additional term in the drift, \(\varepsilon \lambda\), appears because decline in the number of firms \(Q\) leads to higher growth of \(y\). Since all prospective entrants observe the common shock and have a uniform expectation about the idiosyncratic shock, they enter industry at the same threshold, which we denote \(\overline{y}\). New entry limits
the growth of the product price that is associated with innovations in systematic component, implying that the process (6) has a reflecting barrier at \( \bar{y} \). Following Caballero and Pindyck (1992), we assume that the reflecting barrier does not change over time. A sufficient condition for this is a stationary distribution of the number of entering and exiting firms.

In addition to receiving cash flows (1), each firm has an opportunity to expand by paying a fixed cost \( I \), which increases the firm’s output from one unit to \( 1 + \gamma \) units, \( \gamma > 0 \). The assumption of a single growth option is relaxed in Section 3. We assume that investment is irreversible and indivisible, which guarantees that the option value is nonlinear in the profitability shock. Regardless of whether the firm undertakes the investment, it is required to pay operating costs \( I_0 \) at time \( T \). These costs, which are present in most hedging models, create an incentive for a firm to reduce the risk of its cash flows. Such costs can arise because of firm’s obligations, such as employee wages, rents, and other liabilities that are not contingent on the firm’s investment strategies. Firms are different in terms of their initial asset risk (safe or risky project) at the time they enter industry. To keep the analysis simple, we assume that each component of the risk can be high or low, \( \sigma_x \in \{\sigma_{xH}, \sigma_{xL}\} \), \( \sigma_y \in \{\sigma_{yH}, \sigma_{yL}\} \).

The firm can hedge the resulting cash flows generated by these assets. For the ease of exposition, we assume that there is no cost associated with hedging of the systematic component of risk. This assumption is motivated by the fact that inexpensive hedging solutions are routinely accomplished with derivatives linked to fundamentals, such as index prices, foreign currency rates, or commodity prices. Although it becomes increasingly possible to hedge almost any kind of risk with custom derivatives (one example is weather futures), these solutions are costly and cannot eliminate all sources of risk. We refer the reader to Bolton, Chen, and Wang (2009) for an example of a hedging technology with transaction
Since process $y$ has an upper reflecting barrier, the asset with the value $y$ is not tradable and cannot be purchased to hedge other assets. If such an asset were available on the market, it would be possible to construct an arbitrage strategy with a short position in the asset near the upper reflecting boundary. The strategy would generate a positive profit with probability one because $y$ is guaranteed to decrease after reaching the barrier. Therefore, we assume that the firm hedges by buying a correlated futures contract on the industry index. Note that the imperfect correlation with the hedging asset serves as an implicit cost of hedging. Specifically, there is a separate traded financial asset with value $H$ that is correlated with the systematic component of product price

$$dH = H \sigma_H dz_H,$$

$$E[dz_y dz_H] = \theta dt,$$

where $\theta$ measures the contemporaneous correlation between the two assets. To hedge its cash flows, the firm invests in a dynamically managed portfolio consisting of the asset $H$ and a risk-free asset $B$ with constant rate of return $r$,

$$dB = rB dt.$$

The firm’s initial cash reserves that can be used for hedging purposes and investment are $W_0$. The cash holdings available for investment at a later date, $W_T$, depend on the initial cash reserves, the complete history of the cash flow shocks, investment costs and the costs of external financing, and the performance of the hedging portfolio. Note that if it were possible to hedge all cash flow risk with tradable assets, the firm could design a hedging portfolio in a way so as to always have a constant amount of cash available for payment of
operating costs $I_0$.

To justify the need for financial hedging, we assume that the firm is financially constrained and faces convex costs of raising additional capital whenever the firm’s funds are insufficient to cover investment costs. In particular, if the current cash holdings are $W_t$ and the investment cost is $I$, the financing costs are given by a convex function of the cash shortfall,

$$c(I, W_t) = \begin{cases} \max (k (I - W_t)^v, 0), W_t > 0 \\ kI^v, W_t < 0 \end{cases} ,$$

where $v > 0$. By modelling costs in such a way, we assume that the firm is penalized only for a shortfall in cash holdings. It is possible that there are also cost associated with carrying extra cash (e.g., agency problems) that would result in a more symmetric cost function. Our results are qualitatively unaffected if we assume a nonlinear or symmetric cost function.

### 2.1 The Case Without Competition

To analyze the interaction between real investment and financial hedging in a simple setting, we first focus on the case without competition. Specifically, we consider a representative firm that faces a single source of uncertainty $Y$ and operates in the industry with a fixed number of firms, $Q = \mathcal{Q}$, so that the price of output is proportional to the value of the shock

$$P = Y\mathcal{Q}^{-\varepsilon} .$$

Using a contingent claims approach to value the firm’s assets (e.g., Merton (1973) or Leland (1994)), we derive the “continuation value” function for the firm that exercised its investment option. We also obtain the general solution for
the value function prior to the option exercise. Subsequently, the two functions are matched at the point of optimal investment threshold.

Prior to the exercise of the investment option, the firm produces one unit of output. It is well known that the value of the firm $V$ can be found from the ordinary differential equation (ODE)

$$rV = YQ^{-y} + V_y Y + \frac{1}{2} V_{yy} \sigma_y^2 Y^2,$$  \hspace{1cm} (11)

Using the general solution to (11) and noting that the value of the firm must be finite when $Y \to 0$, we obtain

$$V(Y) = \frac{YQ^{-y}}{r - \mu_y} + DY^{b_2},$$  \hspace{1cm} (12)

where $b_2$ is the positive root of the quadratic equation

$$b^2 \sigma^2 + b(2\mu_y - \sigma_y^2) - 2r = 0.$$  \hspace{1cm} (13)

The first term in (12) is the value of the assets in place, whereas the second term has a convenient interpretation of the value of the option to invest in additional capacity. To relate the risk of the firm to its systematic component, we follow Carlson, Fisher, and Giammarino (2004), to compute the firm’s “beta” as

$$\frac{dV(Y)}{dY} \frac{Y}{V(Y)} = 1 + (b_2 - 1) \frac{DY^{b_2}}{V(Y)},$$  \hspace{1cm} (14)

We can separate out two effects of asset risk on beta. The first term in (14) is equal to one because the firm with purely systematic risk and no growth options has the same risk as a market asset. The second term is positive and appears because growth options are more sensitive to the systematic demand shocks than are assets in place.
The value of the firm that has exercised its growth option and has cash flows per unit of time of \((1 + \gamma) Y Q^{-\varepsilon}\), assuming \(r > \mu_y\), is

\[
\hat{V} (Y) = \frac{(1 + \gamma) Y Q^{-\varepsilon}}{r - \mu_y}.
\]  

(15)

At the time of the exercise, the value of the firm is equal to the value after the exercise, minus the investment cost (the value-matching condition)

\[
V (Y^*) = \hat{V} (Y^*) - I.
\]  

(16)

In addition, we require that the time of the exercise is optimal. The necessary conditions for the optimal exercise, identical to the direct maximization with respect to stopping time, is that the first derivatives of \(\hat{V}\) and \(V\) are equal at the exercise threshold. This is known as a smooth-pasting or high-contact condition (Dumas (1991) and Dixit (1993))

\[
V_Y (Y^*) = \hat{V}_Y (Y^*).
\]  

(17)

Using these conditions, it is straightforward to show that firm value prior to the exercise, \(V (Y)\), and the threshold for the optimal option exercise, \(Y^*\), are given, respectively, by

\[
V (Y) = \frac{Y Q^{-\varepsilon}}{r - \mu_y} + \left(\frac{\gamma Y^* Q^{-\varepsilon}}{r - \mu_y} - I\right) \left(\frac{Y}{Y^*}\right)^{b_2}
\]  

and \(Y^* = \frac{b_2 (r - \mu_y) I}{(b_2 - 1) \gamma Q^{-\varepsilon}}\).

(18)

The first part of (18) captures the value from production of one unit of output forever. The second part is the value of the investment option, which is the value of production of additional \(\gamma\) units of output when the price of output
reaches \( Y^*Q^{-\varepsilon} \), minus the investment cost \( I \), all multiplied by the probability that investment option is exercised, \( \left( \frac{Y}{Y^*} \right)^{b_2} \).

Denoting the firm’s cash holdings by \( W_t \) and the fraction of cash invested in the hedging asset by \( \phi \), we can write the dynamics of the firm’s cash reserves as

\[
dW_t = y dt + W_t (1 - \phi) r dt + \phi W_t \sigma_H dz_H, \tag{20}
\]

where \( r \) is the rate at which interest accrues on the amount of cash, \( W_t (1 - \phi) \), invested in the risk-free asset. We follow real options literature (e.g., Zhang (2005), Cooper (2006), and Aguerrevere (2009)) by assuming that the cost of raising financing has no effect on the timing of investment. This assumption implies, for example, that the threshold for optimal investment depends on the realization of the aggregate demand, but is independent of the accumulated cash holdings, \( W_t \).

The firm chooses the optimal hedging ratio by maximizing the expected value at the time of the investment net of the financing costs

\[
\max \frac{Y^*Q^{-\varepsilon}}{r - \mu_y} + \left( \frac{\gamma Y^*Q^{-\varepsilon}}{r - \mu_y} - I \right) \left( \frac{Y}{Y^*} \right)^{b_2} - I_0 e^{r(T-t)} \tag{21}
\]

\[
- E_t[1(t = t^*) e^{r(t-t^*)} (I, \tilde{W}_{t^*}) + e^{r(t-T)} (I_0, \tilde{W}_T)],
\]

where \( W_{t^*} \) is the cash holdings at the time of the investment, and \( W_T \) is cash holdings when the operating costs are paid, and \( 1(t = t^*) \) is the indicator function for investment.

From (21) we observe two results. First, although cash flow risk depends on both volatility \( \sigma_y \) and hedging ratio \( \phi \), the effects of these two parameters on the expected value of the firm are not identical. The value of the growth option (the second term) increases in volatility, but it is independent of the hedging
Second, it follows from (21) that a lower asset risk $\sigma_y$ results in a higher hedging ratio $\phi$. To see this note that the maximzation over $\phi$, keeping the volatility fixed, is equivalent to minimization of the financing costs. Denoting $f(y) \equiv 1 (t = t^*) e^{r(t-t^*)}$,

$$
\min_{\phi} E_t \left[ f(y)c \left( I, \tilde{W}_t \right) + e^{r(t-T)}c \left( I_0, \tilde{W}_T \right) \right] = \\
\min_{\phi} \text{min} \text{cov} \left( f(y), c \left( I, \tilde{W}_t \right) \right) \text{ + } \text{min} \text{cov} \left( f(y), c \left( I, \tilde{W}_t \right) \right) \text{ + } e^{r(t-T)}E \left( c \left( I_0, \tilde{W}_T \right) \right),
$$

where we used definition of covariance. Note that $\tilde{W}$ depends on $\phi$ and appears in three terms. The term $E \left( c \left( I_0, \tilde{W}_T \right) \right)$ is behind the reason for hedging, since $E \left( c \left( I_0, \tilde{W}_T \right) \right) > c \left( I_0, EW_T \right)$ for $c''(.) < 0$. Similarly, term $E \left( c \left( I, \tilde{W}_t \right) \right)$ induces hedging, but its effect on the optimal hedging ratio is smaller because investment is made when $\tilde{W}_t$ is high. Finally, covariance

$$
\text{cov} \left( f(y), c \left( I, \tilde{W}_t \right) \right) < 0
$$

Since financing costs (see (9)) are the function of $(I - W_t)$ the minimization of costs is achieved by minimizing variance of the wealth and maximizing the covariance between wealth and cost of investment. Hedging minimizes the variance of wealth, however it decreases the covariance since probability of investment, $(\frac{Y}{Y^*})^{b^*}$, is increasing with $Y$. We will return to this intuition when we discuss the simulations results.

Next, we focus our attention on the type of the risk. In particular, we introduce competition into the model to create a distinction between the systematic risk and the firm-specific risk.
2.2 The Case with Competition

Competition erodes the value of growth options. When product price increases, for example because of increased demand, investment in additional capacity becomes more valuable for all firms in the industry. However, as firms in industry start to exercise options to increase capacity, and as new entry takes place, the increase in the combined industry output depresses the product price, reducing profit margins.

Suppose firm’s profit follows (1). After the firm exercises its growth option the cash flows per unit of time increase to \((1 + \gamma) (x_i + y)\). Therefore, the value of the firm after exercise is

\[
\hat{V}(x_i, y) = E \int_0^\infty (1 + \gamma) (x_i + y) e^{-(r + \lambda)} dt, \tag{23}
\]

and, similarly to the case discussed above, can be found from a partial differential equation

\[
\hat{r} \hat{V} = (1 + \gamma) (x_i + y) + \hat{V}_x \mu_x x_i + \hat{V}_y (\mu_y + \varepsilon \lambda) y + \frac{1}{2} \hat{V}_{xx} \sigma_x^2 x_i^2 + \frac{1}{2} \hat{V}_{yy} \sigma_y^2 y^2, \tag{24}
\]

where \(\hat{r} = r + \lambda\). To solve equation (24), we invoke our additivity assumption for the cash flows to separate the value due to the firm-specific and systematic shocks. Consider a trial solution

\[
\hat{V}(x_i, y) = v_1(x_i) + v_2(y). \tag{25}
\]

Since equation (24) is separable in variables \(x_i\) and \(y\), it is straightforward to see that the general solution to (24) is equal to the sum of the ODE solution
for \( v_1(x_i) \) and the ODE solution for \( v_2(y) \)

\[
\widetilde{V}(x_i, y) = \frac{(1 + \gamma)x_i}{\bar{r} - \mu_x} + \frac{(1 + \gamma)y}{\bar{r} - \mu_y - \varepsilon \lambda} + Ay^{b_2},
\]

(26)

where \( b_2 \) is the positive root of the quadratic equation

\[
b^2 \sigma^2_y + b (2\mu_y + 2\varepsilon \lambda - \sigma^2_y) - 2\bar{r} = 0.
\]

(27)

The last term in (26) is negative as a result of the limiting effect of competition on the price of output. To prevent the arbitrage we require that when the value of the shock \( y \) approaches the barrier, it must be that

\[
\widetilde{V}_y(x_i, \bar{y}) = 0.
\]

(28)

Prior to the exercise of the option, the value of the firm is

\[
V(x_i, y) = \frac{x_i}{\bar{r} - \mu_x} + \frac{y}{\bar{r} - \mu_y - \varepsilon \lambda} + By^{b_2} + Cx_i^{d_2},
\]

(29)

where \( b_2 \) is the positive root of (27) and \( d_2 \) is the positive root of the similar equation for \( x \)

\[
d^2 \sigma^2_x + d (2\mu_x - \sigma^2_x) - 2\bar{r} = 0.
\]

(30)

The last two terms in equation (29) represent the value of the option to increase capacity and the value adjustment due to the presence of a reflective barrier. Note that \( C > 0 \), whereas the sign of \( B \) depends on whether the value of growth option increases faster in \( y \) than the value erodes due to competition.

Now we discuss how firms exercise their growth options. Intuitively, the option to expand is exercised when both the systematic shock, \( y \), and the firm-specific shock, \( x_i \), are high. Further, given a particular realization of \( y \), there is
a threshold \( x_i^* (y) \), which justifies the irreversible investment. Conversely, given a realization of \( x_i \), there may be a threshold \( y^* (x_i) \). Note that because \( y \) is capped due to the effect of competition, the investment threshold \( y^* \) may not exist when firm-specific shocks \( x_i \) are low.

At the time of the exercise, the value of the firm is equal to the value after the exercise minus the investment cost (the value-matching condition)

\[
V (x_i^*, y^*) = \hat{V} (x_i^*, y^*) - I. \tag{31}
\]

In addition, for the option exercise to be optimal the smooth-pasting condition on the first derivatives has to be satisfied

\[
V_x (x_i^*, y^*) = \hat{V}_x (x_i^*, y^*), \tag{32}
\]
\[
V_y (x_i^*, y^*) = \hat{V}_y (x_i^*, y^*). \tag{33}
\]

Using (28) and (31)-(33), we find constants \( A, B, \) and \( C \) and substitute them in the value functions after and before exercise

\[
\hat{V} (x_i, y) = \frac{(1 + \gamma) x_i}{\hat{r} - \mu_x} + \frac{(1 + \gamma) y}{\hat{r} - \mu_y - \varepsilon \lambda} - \frac{(1 + \gamma) y}{(\hat{r} - \mu_y - \varepsilon \lambda) b_2} \left( \frac{y}{y^*} \right)^{b_2}, \tag{34}
\]
\[
V (x_i, y) = \frac{x_i}{\hat{r} - \mu_x} + \frac{y}{\hat{r} - \mu_y - \varepsilon \lambda} + \frac{\gamma y^*}{(\hat{r} - \mu_y - \varepsilon \lambda) b_2} \left( \frac{y}{y^*} \right)^{b_2} \tag{35}
\]
\[
- \frac{(1 + \gamma) y}{(\hat{r} - \mu_y - \varepsilon \lambda) b_2} \left( \frac{y}{y^*} \right)^{b_2} + \frac{\gamma x_i^*}{(\hat{r} - \mu_x) d_2} \left( \frac{x_i}{x_i^*} \right)^{d_2}.
\]

The thresholds for exercise \( x_i^* \) and \( y^* \) are defined then by the following linear equation

\[
\frac{\gamma x_i^*}{\hat{r} - \mu_x} \frac{d_2 - 1}{d_2} + \frac{\gamma y^*}{\hat{r} - \mu_y - \varepsilon \lambda} \frac{b_2 - 1}{b_2} = I. \tag{36}
\]
Note that when \(x_i\) is sufficiently small, i.e., when

\[
x_i < \frac{I - \frac{\gamma \pi}{r - \mu_y} - \frac{b_2 - 1}{b_2}}{\frac{\gamma}{r - \mu_x} - \frac{d_2 - 1}{d_2}},
\]

there is no optimal threshold for \(y\) which warrants the investment.

Assuming that entry into the market is competitive (as in Leahy (1993) and Caballero and Pindyck (1996)), we can find the entry threshold, \(\bar{y}\), by requiring that the expected profit at entry is zero

\[
V(x_0, \bar{y}) = R,
\]

(37)

where \(x_0\) is the initial expected draw of \(x_i\) and \(R\) is the cost of entry. We follow Caballero and Pindyck (1996) in assuming that the threshold for the firm entry is independent of the number of firms that have entered in the past.

The dynamics of firm’s cash holdings \(W_t\) can be expressed as (we suppress subscript \(t\) for all variables except \(W_t\))

\[
dW_t = (x_i + y) dt + \phi W_t \sigma_H dz_H + W_t (1 - \phi) r dt,
\]

(38)

where \(r\) is the rate at which interest accrues on the remaining cash, \(W_t (1 - \phi)\), that is invested in the risk-free asset.

The firm chooses the optimal hedging ratio by maximizing the expected value at the time of the investment net of the financing costs

\[
\max_{\phi} V(x_i, y) - I_0 e^{\gamma(T-t)} - \Pr(Inv)c(I, W_t) - e^{\gamma(T-t)}c(I_0, W_T), \tag{39}
\]

where the value of future cash flows from the assets, \(V(x_i, y)\), is independent of the hedging policy, \(\phi\), and \(\Pr(Inv)\) is the today’s value of the contingent claim.
that pays $1 conditional on the investment; it depends on $x$ and $y$.

From (39), one can see that the incentives to hedge are lower for the firms with larger idiosyncratic component of risk. Intuitively, the value $V(x_i, y)$ is less sensitive to the systematic component because of the product market competition. The value of the growth option is more sensitive to the firm-specific shock $x_i$. Therefore, firm value decreases in $\rho$ (from (1)) and the book-to-market ratio increases in $\rho$. Our results indicate that growth firms have a smaller incentive to engage in financial hedging. These implications are consistent with empirical evidence that large profitable firms with fewer growth opportunities tend to hedge more (Mian (1996) and Bartram, Brown, and Fehle (2009)).

Next, we turn to the equity betas and stock returns. Since $y$ represents the aggregate uncertainty in the model, we define firm’s equity beta as the elasticity of the firm market value with respect to the systematic factor $y$. Following, Carlson, Fisher, and Giammarino (2004) and Aguerrevere (2009), the firm $i$’s beta is

$$\beta_i = \frac{dV(x_i, y)}{dy} \frac{y}{V(x_i, y)},$$

(41)

From (35), we can write

$$\beta_i = 1 - \frac{V^U(x_i)}{V(x_i, y)} + \frac{V^G(y)}{V(x_i, y)} - \frac{V^C(y)}{V(x_i, y)},$$

(42)

where $V(x_i, y)$ is given by (35) and expressions for $V^U(x_i)$, $V^G(y)$, and $V^C(y)$ are in the Appendix.

The first term in (42) is equal to one because of the normalization. The second term appears because part of firm value is derived from the unique assets that are uncorrelated with the aggregate demand uncertainty and thus reduce the overall firm’s exposure to the systematic risk. The third term denotes the increase in the firm’s risk due to the presence of growth options since options are
more sensitive to the aggregate uncertainty than are assets in place. Finally, the fourth term appears because of the limiting effect of competition on the value of growth options and firm’s cash flows (Aguerrevere (2009)). Note that it is possible for the net effect of the last two terms to be either positive or negative depending on the size of the firm’s growth options, the industry entry threshold, and the optimal investment expansion threshold.

It follows from (42) that firms with more unique assets, have more valuable growth options since competition has no attenuating effect on the value of assets derived from the firm-specific components. At the same time, firms with more unique assets have smaller betas. This implies that growth firms have lower expected returns in the cross-section.

It is also useful to calculate the firm’s beta after it has invested in the expansion. From (34)

\[ \hat{\beta}_i = 1 - \frac{\tilde{V}U(x_i)}{V(x_i, y)} - \frac{V^C(y)}{\hat{V}(x_i, y)} \]  

(43)

where \( \hat{V}(x_i, y) \) is given by (34) and expression for \( \tilde{V}U(x_i) \) is in the Appendix. It is easiest to compare the betas (42) and (43) exactly at the point of option exercise since then \( \tilde{V}U(x_i^*) = V^U(x_i^*) \) and from (31) there is a simple relation between \( V \) and \( \hat{V} \). First, note that, similar to the effect described by Carlson, Fisher, and Giammarino (2006), the systematic risk becomes lower after option exercise because growth options are generally more sensitive to the aggregate shocks than are assets in place. In particular, the term \( \frac{V^C(y)}{\hat{V}(x_i, y)} \) disappears after exercise. There is also an offsetting effect on beta since \( \hat{V} > V \), and this decreases the importance of terms two and three in (43).

Next, consider betas away from the point of optimal exercise. It is easy to see that since all firms observe the same aggregate demand shock \( y \), those that have exercised their growth options must have had larger realizations of the
firm-specific demand shocks. This implies that in the cross-section, firms that have exercised their options, also have a lower systematic risk.

<table>
<thead>
<tr>
<th>Firms have not exercised options ((B = \frac{L}{\gamma}))</th>
<th>Firms have exercised options ((B = \frac{L}{\gamma} + I))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low (x_i)</td>
<td>Medium (x_i)</td>
</tr>
<tr>
<td>Low (x_i) (\rightarrow) low value of options (\rightarrow) low (V).</td>
<td>Medium (x_i) (\rightarrow) medium value of options (\rightarrow) medium (V).</td>
</tr>
<tr>
<td>Low (x_i) (\rightarrow) high (\beta).</td>
<td>Medium (x_i) (\rightarrow) medium (\beta).</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The table above summarizes the effects of firm-specific demand shocks on the book-to-market ratio and the systematic risk in the model. \(B\) denotes the book value of assets. To be consistent with our earlier assumption that installing \(\gamma\) units of capacity costs \(I\), we assume that the cost of one unit is \(\frac{L}{\gamma}\). Since all firms have one unit of capacity prior to option exercise, their book value can be expressed as \(B = \frac{L}{\gamma}\). Given that firms have no leverage in our model, the book-to-market equity value is measured in the model by \(\frac{B}{V}\).

Next, we generalize the model to the case with infinite number of options.
3 The Model with Infinite Number of Investment Options

This section extends the results by considering a more general case with continuous investment (infinite number of options) and continuously paid operating costs. Firms are assumed to be price takers. Revenues depend on the systematic and idiosyncratic profitability shocks and are generated by a production function with decreasing returns to scale. The assumption of decreasing returns to scale is important because otherwise, in the case with infinite number of options, there will be an incentive to invest unlimited amount at a higher profitability. In the interest of making a clear presentation, we fix the number of the firms in the industry and re-consider this assumption later. There are operating costs which are proportional to the capital size. Such costs create the need for external financing whenever revenues decrease. We assume that financing is costly. In particular, there are nonlinear costs of financing which apply whenever revenues fall below the costs. The firm can invest irreversibly, incrementally, and without fixed costs, but subject to the constant marginal price of each unit of installed capital. Changes in cash reserves of the firm are equal to the profits net of the costs of investment. The firm pays separate financing costs when cash is not sufficient to cover investment expense.

Following these assumptions, the firm’s instantaneous operating profit is given by

$$\pi_t = K_t^\gamma (Y_t + X_{it}) - mK_t,$$

(44)

where $\gamma < 1$, $K$ is the installed capital, and $Y$ and $X$ are the systematic and idiosyncratic demand shocks, respectively. The shocks $Y$ and $X$ follow a
geometric Brownian motion process (in the risk-neutral measure)

\[
\begin{align*}
\frac{dX_t}{X_t} & = \mu_X dt + \sigma_X \left( \sqrt{1 - \rho_x^2} dz_x + \rho_x dz_H \right), \quad (45) \\
\frac{dY_t}{Y_t} & = \mu_y dt + \sigma_y \left( \sqrt{1 - \rho_y^2} dz_y + \rho_y dz_H \right). \quad (46)
\end{align*}
\]

We allow for the possibility that both types of shocks are correlated with the hedging asset. As a particular case, one could consider the scenario in which \( \rho_x = 0 \), which is relevant when only the systematic component of the asset can be hedged. The dynamics of the hedging asset’s value is also described by a geometric Brownian motion

\[
\frac{dH_t}{H_t} = \sigma_H dz_H \quad (47)
\]

To reduce the volatility of the operating profits, the firm invests cash \( S_t \) in the hedging portfolio, with a fraction \( \phi \) invested in a hedging asset, and \( 1 - \phi \) in the risk-free asset. The proceeds from the hedging portfolio are

\[
\frac{dS_t}{S_t} = (1 - \phi) r dt + S_t \phi \sigma_H dz_H, \quad (48)
\]

where \( r \) is the rate of return on the risk-free asset.

The total profit of the firm, including proceeds from the hedging portfolio, is subject to costs

\[
CF = \pi_t + dS_t - k_1 (\pi_t + dS_t)^{v}, \quad (49)
\]

where \( k_1 > 0 \) is constant. In contrast to the single option case in the previous section, we assume that the costs of financing the shortfalls in cash flows and costs of financing investment are different.

We assume that there are no fixed costs of investment, whereas the marginal
cost of installing an additional unit of capacity is $\alpha$, so that the direct price of investment is $adK$. Analogously to the single option case, we assume that whenever the firm does not have sufficient funds for the required investment, it raises funds externally and incurs external financing costs

$$c(dK, W_t) = \begin{cases} k_2 \max(\alpha dK - W_t, 0)^\nu, & \text{if } W_t > 0 \\ k_2 (\alpha dK)^\nu, & \text{if } W_t \leq 0 \end{cases}$$

(50)

where $k_2 > 0$ is constant. Here we assume that the external financing comes in the form of additional borrowing, which implies that in our model it is possible for the firm to have negative cash $W_t$. Whenever $W_t$ is negative, we compute the financing cost on the total amount of investment. The change in firm’s cash holdings is then

$$W_t - W_{t-1} = CF_t - \alpha dK - c(dK, W_t)$$

(51)

To keep the model tractable, we assume that the investment policy is independent of the financing policy (for example, relying on the Modigliani-Miller assumptions), and therefore develop the optimal investment model with irreversibility being the only friction.

We model product market competition by assuming that new firms can enter the industry by paying a cost $R$. Whereas the value of shock $Y$ is common knowledge prior to entry, the prospective entrants cannot observe the values of their idiosyncratic shocks until they pay an entry cost and get a random draw of $X_i$. For simplicity, we adopt the assumption that firms in the market are identical, competitive, risk neutral, and infinitesimally small. The last assumption allows us to treat firms as price-takers and to ignore the effect of firm’s own output on the equilibrium price.

Since all prospective entrants observe the common shock and have a uni-
form expectation about the idiosyncratic shock, they enter industry at the same
threshold, which we denote $\overline{Y}$. New entry limits the growth of the product price
that is associated with innovations in systematic component, implying that the
process (6) has a reflecting barrier at $\overline{Y}$.

The solution for optimal investment follows Pindyck (1988). The value of
the firm in the inaction region

$$V(Y, X_i, K) = B(K)Y^\beta + A(K)X_i^b + \frac{(Y + X_i)K^\gamma}{r - \mu} - \frac{mK}{r} \quad (52)$$

where $\beta > 0$ is a root of the usual quadratic equation and is given by

$$\beta = \frac{1}{2} - \frac{\mu_y}{\sigma_y^2} + \sqrt{\left(\frac{\mu_y}{\sigma_y^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma_y^2}} \quad (53)$$

$$b = \frac{1}{2} - \frac{\mu_x}{\sigma_x^2} + \sqrt{\left(\frac{\mu_x}{\sigma_x^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma_x^2}} \quad (54)$$

The first two terms in (52) reflect the value of real options to increase capital
in the future as well as the limiting effect of competition. To find the constants
$B(K), A(K), C(K)$ and the optimal amount of installed capital $K^*$, we use
the boundary condition for the investment

$$\frac{\partial V}{\partial K} |_{K^*} = \kappa \quad (55)$$

and the smooth-pasting conditions at the boundary

$$\frac{\partial}{\partial Y} \left[ \frac{\partial V}{\partial K} \right] |_{K^*} = 0 \quad (56)$$

$$\frac{\partial}{\partial X_i} \left[ \frac{\partial V}{\partial K} \right] |_{K^*} = 0 \quad (57)$$

27
The solution follows

\[ V_K = B'(K^*) Y^\gamma + A'(K^*) X^b + \frac{(Y + X) \gamma K^{\gamma - 1}}{r - \mu} - \frac{m}{r} = \alpha \] (58)

\[ V_{KY}|_{K^*} = \beta B'(K^*) \gamma Y^{\gamma - 1} + \frac{\gamma K^{\gamma - 1}}{r - \mu} = 0 \] (59)

\[ V_{KX}|_{K^*} = b A'(K^*) X^{b-1} + \frac{\gamma K^{\gamma - 1}}{r - \mu} = 0 \] (60)

Solving this gives

\[ A'(K^*) = -\frac{\gamma K^{\gamma - 1} X^{1-b}}{(r - \mu)^b} \] (61)

\[ B'(K^*) = -\frac{\gamma K^{\gamma - 1} Y^{1-\beta}}{(r - \mu)^\beta} \] (62)

Substituting

\[ K^* = \left( \frac{1}{\gamma} \left( \frac{\alpha + m}{(r - \mu)} \right) \frac{1}{(r - \mu)^b} \right)^{\frac{1}{\gamma - 1}} \] (63)

Integrating the constants gives

\[ A(K) = \int_K^\infty \frac{\gamma K^{\gamma - 1} X^{1-b}}{(r - \mu)^b} dK = \frac{K^{\gamma X^{1-b}}}{(r - \mu)^b} \] (64)

\[ B(K) = \frac{K^{\gamma Y^{1-\beta}}}{(r - \mu)^\beta} \] (65)

Substituting \( V \) in these two conditions produces values for \( B'(K) \) and the optimal installed capital \( K^*(y) \)

\[ B'(K) = -\left( \frac{\beta_1 - 1}{\alpha + m} \right)^{\beta_1 - 1} \left( \frac{\gamma}{(r - \mu) \beta_1} \right)^{\beta_1} K^{\beta_1 (\gamma - 1)} \] (66)

\[ K^* = \left( \frac{(\beta_1 - 1) \gamma Y}{\beta_1 (\alpha + m) (r - \mu)} \right)^{\frac{1}{\gamma - 1}} \] (67)
Integrating the first equation yields the function \( B(K) \)

\[
B(K) = \int_{K}^{\infty} (-B'(K)) \, dK = \left( \frac{\beta_1 - 1}{\alpha + \frac{\alpha \gamma}{r}} \right)^{\beta_1 - 1} \left( \frac{\gamma}{(r - \mu) \beta_1} \right)^{\beta_1} \frac{K^{\beta_1 (\gamma - 1) + 1}}{\beta_1 (\gamma - 1) + 1}
\]

(68)

Here we have used the limiting Inada condition that

\[
K^{\beta_1 (\gamma - 1) + 1} |_{\infty} = 0
\]

(69)

which implies that for convergence we need \( \beta_1 > \frac{1}{1 - \gamma} \).

Consider separately the limiting effect of the competition. To prevent arbitrage it must be that at the reflective barrier \( \gamma \)

\[
V_Y(X_i, \gamma) = 0.
\]

(70)

Note that this condition does not affect the investment policy.

The case with infinite number of growth options and constant incentives for hedging is instructive because it demonstrates that the effects described in the previous section using an example with a single option are important and comparable in magnitude to the value of the firm. More important, the incremental investment case allows us to overcome some obvious limitations of the single option case. For example, in the former case the amount of investment is correlated with profitability shock, but in the latter it is fixed by assumption.

4 Simulation Results and Empirical Implications

In this section, we rely on the simulations to expose the intuition from the model. While it is not the main objective to use the model for simulations, it seems appropriate to ensure that the model’s basic implications are reasonable before examining the effects the properties of returns produced by the model.
To achieve this, we choose the case with a single investment option and a single operating cost. We further restrict the time of the exercise of the option to be the same as the time of the payment of the operating costs. The goal is to demonstrate that the results hold purely due to the positive correlation between the investment demand and the cash accumulation.

The values of the model parameters used in the simulations are as follows. Initial wealth $W_0$ is 1000 to ensure that the firm arrives to investment over- and under-financed with approximately equal probabilities. Initial value of the shock, $y_0$, is normalized to 5 and the drift is set to 1 percent, which is dictated by the requirement that the growth rate of cash flows must be smaller than the discount factor for the value of the assets to be finite. We set the base volatility to 0.2 to match the annual volatility of the market of about 20%. The operating costs are 1000; costs must be comparable to the level of initial wealth to ensure that at least some hedging is optimal. The expansion option is assumed to have a parameter $\gamma = 1$, which means that the cash flows are doubled at the time of the exercise; the exact value of this parameter has no implication on the results, it simply results in the delay or acceleration of the exercise. We vary financing costs of raising external capital to ensure that the costs, on expectation, are comparable to the value of firms assets; the base case uses costs of just 0.1%. Within the simulation, we gradually adjust the hedging ratio parameter, $\phi$, from the minimum (0) to maximum value (1) and repeat the procedure 1,000 times for each value of this parameter, averaging the results over the repetitions.

Further, to evaluate our basic model’s ability to reproduce some of the key features of returns data, we attempt to match the dynamics of the cross-section of firms using the panel of data simulated from the model. To make our results more comparable to the actual data used in Fama and French (1992), we simulate monthly observations for 2,000 firms over the period of 420 months. To
make sure that the data reaches the stationary distribution we drop the first 60 months for each run. We assume that the investment cost and operating cost is payable after 120 months and verify that choosing a different horizon does not qualitatively affect the results. Using a simulated panel of data, we form 12 portfolios based on the ranked values of book-to-market equity ratios. Portfolios two through nine use the deciles of B/M, whereas portfolios one and ten are each split into half. The portfolios are held from July of year $t$ through the end of June of year $t + 1$, and the time-series average returns are calculated for each portfolio. The book-to-market ratios are calculated using the data from the end of December of year $t - 1$.

Figure 1 and Figure 2 are designed to offer the reader a better understanding of the simulation process. Figure 1 describes a randomly selected path for the shock process $x_i$ and Figure 2 describes the path for $y$. The former is generated anew for each of the simulated firm. The latter is recycled when the data for the next firm is produced to ensure that the systematic component of the profit is kept constant in the cross section. Figure 2 shows that the systematic shock bounces back when it reaches its upper reflective barrier.

Figure 3 gives the investment threshold $x^*(y)$ as a function of time. Each value on this graph represents the threshold value of the firm-specific shock that can justify the investment under the current value of the systematic shock. We observe that the shape of this graph is the mirror image of the systematic shock. This is because the two components of the profits are modeled as additive. Therefore, a higher value for one component means that a lower value for the other component can still make the investment optimal.

In Figure 4, we plot the evolution of the number of the firms in the industry $Q$. The smooth downward adjustments in the number of firms are due to the gradual decay of the firms, while the discontinuous upward jumps are due to
the entry of new firms. Notice that because lower values of $Q$ make entering the market more attractive, we observe in this simulation a large number of firms entering in the second part of the graph. The resulting value of the firm $V$ is plotted in Figure 5; this value does not include the cost of investment or financing.

The evolution of the firm’s market beta is shown in Figure 6. The graph illustrates that beta changes because of the two effects. When there is no investment, firm’s beta increases with the ratio of the systematic risk shock to the firm specific risk shock. This first effect is visible as the smooth adjustments on the graph. At the time of the investment, beta experiences a discontinuous downward jump because some of the growth options are converted into assets in place, which have a lower sensitivity to the value of the shock.

Figures 7, 8, and 9 show the relation between beta, book-to-market ratio, and average returns. The purpose of these graphs is to show that, although the beta in the model completely determines sensitivity of the firm value to the firm, the relation between betas and returns is not one-to-one. This is because in the model the exact relation holds only conditionally, but we use the average values on the graphs. Figure 7 shows that the relation between the stock returns and the book-to-market equity ratios is close to linear. Figure 8 shows the direct relation between the conditional beta and book-to-market in the model. Figure 9 displays the relation between the expected stock returns and the unconditional beta of the portfolio. It can be seen that when beta is measured with error the relation between the expected returns and the book-to-market ratio may be stronger than the one between expected returns and proxies for beta.

Figures 10 and 11 show that, in line with our intuition, the operating costs are reduced with hedging; however the costs of contingent investment tend to increase in the hedging ratio. The total costs of raising external finance are
minimized at an interior hedging ratio since this allows for a tradeoff between operating costs and the costs associated with contingent investment (see Figure 12).

Table 1 additionally explores the difference between the data produced by our model and the actual data. Panel A reports the results from Fama and French (1992) using the historical data. Panel B reports the results based on the data simulated from the model. The average stock returns in our data increase as the $\log(B/M)$ ratio increases. Note that the average magnitude of returns is lower in the simulated data than in the historical data because our model is not designed to explain the market premium. On the other hand, Table 1 shows that our model produces a significant dispersion in betas across the portfolios—a phenomenon absent in Fama and French (1992). The latter result is not surprising because in our (one-factor risk) model the book-to-market ratios are related to returns only through beta, and also because we can measure betas precisely in the simulations while the plots use the average values.

5 Conclusion

According to executives, corporate risk management is a strategic activity that can encompass anything from operating changes to financial hedging to the buying and selling of plants and new businesses. Our paper focuses on the interaction of real hedging (ability to choose safer assets) and financial hedging (use of financial instruments to lower the risk of cash flows) in the environment where firms invest under financing constraints.

The main premise of the model is that investing in safer assets is not a substitute for financial hedging since it not only reduces risk but also decreases the value of real options attached to firm’s assets. We show that the optimal
amount of financial hedging trades off the benefits of safer cash flows with ability to finance investment in growth options. Our results indicate that more real hedging may lead to more financial hedging, which implies that safe mature firms may use more financial derivatives.

We also show that, in addition to the magnitude of risk, the type of risk is an important determinant of a firm’s hedging policy. Specifically, growth firms, which are distinguished by a larger proportion of idiosyncratic risk, have a smaller incentive to engage in financial hedging. The intuition for this result is that firms with more unique assets are less affected by product market competition and have more valuable investment options, which implies that growth firms have larger investment demand in states with high cash flow than do value firms. Implications obtained from the model are consistent with empirical evidence that large profitable firms with fewer growth opportunities tend to be large users of financial derivatives. An interesting aspect of this result is that, in contrast to previous studies, it does not hinge on the costs of hedging.

Finally, we show that because of the differential effect of competition on real options of firms with high idiosyncratic and high systematic risk components, there is a positive relation between book-to-market ratios and equity betas. Our results thus contribute to the explanation of the “value puzzle.”
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Appendix

The value of the firm before option exercise is $V(x_i, y)$ and is given by (35).

Firm $i$’s beta is then given by

$$\beta_i \equiv \frac{dV(x_i, y)}{dy} \frac{y}{V(x_i, y)} = 1 - \frac{V^U(x_i)}{V(x_i, y)} + \frac{V^G(y)}{V(x_i, y)} - \frac{V^C(y)}{V(x_i, y)},$$  \hspace{1cm} (71)

where

$$V^U(x_i) = \frac{x_i}{\bar{r} - \mu_x} + \frac{\gamma x_i^*}{(\bar{r} - \mu_x) d_2} \left( \frac{x_i^*}{x_i} \right)^{d_2},$$  \hspace{1cm} (72)

$$V^G(y) = \frac{b_2 - 1}{b_2} \frac{\gamma y^*}{(\bar{r} - \mu_y - \varepsilon \lambda)} \left( \frac{y}{y^*} \right)^{b_2},$$  \hspace{1cm} (73)

$$V^C(y) = \frac{b_2 - 1}{b_2} \frac{(1 + \gamma) \bar{y}}{(\bar{r} - \mu_y - \varepsilon \lambda)} \left( \frac{y}{\bar{y}} \right)^{b_2}. $$  \hspace{1cm} (74)

The firm’s beta after it has invested in the expansion can be obtained from (34)

$$\tilde{\beta}_i \equiv \frac{d\hat{V}(x_i, y)}{dy} \frac{y}{\hat{V}(x_i, y)} = 1 - \frac{\hat{V}^U(x_i)}{\hat{V}(x_i, y)} - \frac{V^C(y)}{\hat{V}(x_i, y)},$$  \hspace{1cm} (75)

where

$$\hat{V}^U(x_i) = \frac{(1 + \gamma) x_i}{\bar{r} - \mu_x}. $$  \hspace{1cm} (76)
Table 1. Properties of portfolios formed on book-to-market.

The table presents returns and betas for the 12 portfolios formed based on ranked values of book-to-market ratios $\log(B_t/V_t)$. Portfolios are rebalanced at the end of June of each year. The break points for the book-to-market ratios are based on ranked values of book-to-market equity ratios as of end of December of previous year. Panel A is reproduced from Fama and French (1992). Panel B uses the simulated data from the base model. The average returns are the time-series averages of the monthly equally-weighted portfolio returns, in percentages; $\log(B/V)$ are the time-series averages of the monthly values of these portfolios, and $\beta$ is the time-series average of monthly portfolio beta.

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Figure 1. The sample path of the firm-specific demand shock $x_i$ as a function of time (in months).

Figure 2. The sample path of the systematic demand shock $y$ as a function of time (in months). The reflecting barrier is set at value of $\bar{y} = 5$. 

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Figure 3. The investment exercise threshold $x^*(y)$ as a function of time (in months).

Figure 4. The number of firms in the industry as a function of time. Firms enter optimally when the systematic demand shock reaches $\bar{y} = 5$. Absent entry, the number of firms declines over time with intensity $\lambda = 0.1$. 
Figure 5. Sample path of firm value as a function of time (in months).

Figure 6. Firm equity beta as a function of time (in months). The dotted time depicts the time of the investment.
Figure 7. The average monthly portfolio stock returns as function of the portfolio’s average book-to-market equity ratio. The portfolios are formed based on deciles of the B/M ratio as of the end of December and held from July to June of the next year. The top and bottom deciles are split into two portfolios.

Figure 8. The average portfolio beta as function of the portfolio’s average book-to-market equity ratio. The portfolios are formed based on deciles of the B/M ratio as of the end of December and held from July to June of the next year. The top and bottom deciles are split into two portfolios.
Figure 9. The average stock returns as function of the portfolio’s average beta.

The portfolios are formed based on deciles of the B/M ratio as of the end of December and held from July to June of the next year. The top and bottom deciles are split into two portfolios.
Figure 10. The costs of raising external capital to finance operating expenses as a function of hedging ratio. The volatility of the systematic shock is \( \sigma_y = 0.2 \), the costs of external financing are \( k = 0.1\% \), the drift is \( \mu_y = 0.01 \), the risk-free rate is \( r_f = 0.04 \), and the operating costs are \( I_0 = 1000 \).

Figure 11. The costs of raising external capital to finance contingent investment as a function of hedging ratio. The volatility of the systematic shock is \( \sigma_y = 0.2 \), the costs of external financing are \( k = 0.1\% \), the drift is \( \mu_y = 0.01 \), and the risk-free rate is \( r_f = 0.04 \).
Figure 12. The total costs of external financing (contingent investment and operating costs) as a function of hedging ratio. The volatility of the systematic shock is set to $\sigma_y = 0.2$, the costs of external financing are $k = 0.1\%$, the drift is $\mu_y = 0.01$, and the risk-free rate is $r_f = 0.04$. 